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Hochschild Cohomology in Algebra, Geometry, and Topology

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Abstract. In 1945 Gerhard Hochschild published *On the cohomology groups of an associative algebra* in the *Annals of Mathematics* and thereby created what is now called Hochschild theory. In 1963, Murray Gerstenhaber proved that the Hochschild cohomology of any associative algebra carries a super-Poisson algebra structure, comprised of a graded commutative cup product and an odd super Lie algebra structure that acts through graded derivations with respect to the product. Subsequently, a number of higher structures have been discovered, and a vast body of research concerning and/or using Hochschild theory has developed in many different fields in mathematics and physics.

Mathematics Subject Classification (2010): 16E, 13D, 14F, 55N, 83E30.

Introduction by the Organisers

This meeting had 27 participants from 10 countries (Argentina[2], Belgium[3], Canada[2], China[3], France[4], Germany[1], Norway[3], Russia[2], UK[1], and the US[6]) and 20 lectures were presented during the five day period. The extended abstracts of these lectures are presented on the following pages in chronological order.

This workshop fostered exchange of knowledge and ideas between various research areas, developed existing collaborations, and identified new directions of research by bringing together leading researchers and young colleagues from Algebraic Geometry (in its classical and its noncommutative version), Singularity Theory, Representation Theory of Algebras, Commutative Algebra, and Algebraic

Topology. The choice of a coherent group of disciplines, rather than a broad coverage of Hochschild theory, allowed for effective communication between different groups of practitioners.

Survey lectures on Hochschild cohomology of algebraic varieties, the relationship between loop homology and Hochschild cohomology in algebraic topology, and on the Hochschild cohomology of block algebras of finite groups were complemented by presentations on higher order structures on Hochschild cohomology such as existence of a Batalin–Vilkovisky operator or the explicit form of the Gerstenhaber Lie bracket in special cases. Further, categorical interpretations of various aspects of Hochschild theory were presented, and variations of Hochschild cohomology such as Koszul or Poisson cohomology were studied.

Numerous discussions among the participants, in particular among participants belonging to different mathematical communities, have contributed to the workshop in an essential way. As always, such workshop at MFO provided an ideal atmosphere for fruitful interaction and exchange of ideas. It is a pleasure to thank the administration and the staff of the Oberwolfach Institute for their efficient support and hospitality.

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Hochschild Cohomology in Algebra, Geometry, and Topology

Table of Contents

Abstracts

Hochschild cohomology of smooth algebraic varieties

Damien Calaque
(joint work with Carlo A. Rossi & Michel Van den Bergh)

Let X be a smooth algebraic variety (over a field of zero characteristic). We define its *Hochschild cohomology ring* to be

$$HH^{\cdot}(X) := Ext^{\cdot}_{X\times X}(\Delta_*\mathcal{O}_X, \Delta_*\mathcal{O}_X)\,,$$

where $\Delta : X \to X \times X$ is the diagonal map.

1. Hochschild cohomology as the (hyper)cohomology of poly-differential operators

1.1. **Local Hochschild cochains.** We have the following sequence of ring isomorphisms:

$$\begin{aligned} Ext^{\cdot}_{X\times X}(\Delta_*\mathcal{O}_X, \Delta_*\mathcal{O}_X) &\cong \mathbb{R}\Gamma\big(X\times X, \mathbb{R}\mathcal{H}om_{\mathcal{O}_{X\times X}}(\Delta_*\mathcal{O}_X, \Delta_*\mathcal{O}_X)\big) \\ &\cong \mathbb{R}\Gamma\big(X, \mathbb{R}\mathcal{H}om_{(\pi_1)_*\mathcal{O}_{X\times X}}(\mathcal{O}_X, \mathcal{O}_X)\big) \\ &\cong \mathbb{R}\Gamma\big(X, \mathbb{R}\mathcal{H}om_{(\pi_1)_*\mathcal{O}_{\widehat{X\times X}}}(\mathcal{O}_X, \mathcal{O}_X)\big)\,, \end{aligned}$$

where π_1 is the first projection and $\widehat{X\times X}$ is the formal neighborhood of the diagonal in $X \times X$. The last identification comes from the fact that $(\pi_1)_*\mathcal{O}_{\widehat{X\times X}}$ is flat over $(\pi_1)_*\mathcal{O}_{X\times X}$.

Below we provide an explicit description of the algebra

$$\mathbb{R}\mathcal{H}om_{(\pi_1)_*\mathcal{O}_{\widehat{X\times X}}}(\mathcal{O}_X, \mathcal{O}_X)\big)$$

of *local Hochschild cochains*, as an algebra object in $\mathrm{D}(\mathcal{O}_X\text{−}mod)$.

1.2. **Local Hochschild cochains as Lie algebroid Hochschild cochains.** Let $\mathcal{L}$ be a Lie algebroid over X which is locally free of finite rank as an $\mathcal{O}_X$-module. As an example to keep in mind, one can consider the tangent Lie algebroid $\mathcal{L} = T_X$. There are several algebraic objects one can associate to $\mathcal{L}$, such as:

- its *universal enveloping algebra* $U(\mathcal{L})$, which is a filtered Hopf algebroid. Whenever $\mathcal{L} = T_X$, $U(\mathcal{L})$ is the algebra of differential operators on X.
- its *jet algebra* $J(\mathcal{L})$, defined as the $\mathcal{O}_X$-linear dual to $U(\mathcal{L})$, and that one can view as the algebra on the formal groupoid integrating $\mathcal{L}$. Whenever $\mathcal{L} = T_X$, $J(\mathcal{L})$ is isomorphic to $(\pi_1)_*\mathcal{O}_{\widehat{X\times X}}$.

  Sketch of proof of this fact. The isomorphism sends a section f of $(\pi_1)_*\mathcal{O}_{\widehat{X\times X}}$ to the jet j_f defined as follows: j_f sends a differential operator P to $(id \otimes P)(f)$, which is a section of $\mathcal{O}_X$ because P has finite order. □

- its *Hochschild cohomology ring* $HH^{\cdot}_{\mathcal{L}} := Ext_{J(\mathcal{L})}(\mathcal{O}_X, \mathcal{O}_X)$.

The upshot is that we can describe the algebra of local Hochschild cochains as

$$\mathbb{R}\mathcal{H}om_{J(\mathcal{L})}(\mathcal{O}_X, \mathcal{O}_X)\,,$$

with $\mathcal{L} = T_X$.

1.3. **An explicit description of Lie algebroid Hochschild cochains.** Borrowing the notation from above, we have the following:

Proposition 1.1 ([2])**.** *There is an isomorphism of algebras*

$$\mathbb{R}\mathcal{H}om_{J(\mathcal{L})}(\mathcal{O}_X, \mathcal{O}_X) \cong (\mathcal{D}^{poly,\cdot}_{\mathcal{L},X})^{op}$$

in $\mathrm{D}(\mathcal{O}_X{-}mod)$*. Here* $\mathcal{D}^{poly,n}_{\mathcal{L},X} := U(\mathcal{L})^{\otimes_{\mathcal{O}_X} n}$*, the product is the concatenation, and the differential is the Cartier (a-k-a co-Hochschild) differential for the coalgebra* $U(\mathcal{L})$*.*

Whenever $\mathcal{L} = T_X$, $\mathcal{D}^{poly,n}_{\mathcal{L},X}(U)$ is the subcomplex of the Hochschild complex of $\mathcal{O}_X(U)$ consisting of these cochains that are differential operators in each argument.

Sketch of proof of the Proposition. Note that $J(\mathcal{L})$ is a topological algebra, and that the morphism

$$\mathbb{R}\mathcal{H}om^{cont.}_{J(\mathcal{L})}(\mathcal{O}_X, \mathcal{O}_X) \to \mathbb{R}\mathcal{H}om_{J(\mathcal{L})}(\mathcal{O}_X, \mathcal{O}_X)$$

is an isomorphism in $\mathrm{D}(\mathcal{O}_X{-}mod)$. Let us now give an explicit resolution $\mathcal{B}^{\cdot}J(\mathcal{L})$ of $\mathcal{O}_X$ as a topological $J(\mathcal{L})$-module:

$$\mathcal{B}^n J(\mathcal{L}) = J(\mathcal{L})^{\hat{\otimes}(n+1)}$$

and the differential sends $j_0 \otimes \cdots \otimes j_n$ to

$$j_0 j_1 \otimes \cdots \otimes j_n + \cdots + (-1)^n j_0 \otimes \cdots \otimes j_{n-1} j_n + (-1)^{n+1} j_0 \otimes \cdots \otimes j_{n-1} j_n(1)\,.$$

We conclude by noting there is a (right) action of $\mathcal{D}^{poly,\cdot}_{\mathcal{L},X}$ on $\mathcal{B}^{\cdot}J(\mathcal{L})$. □

2. Hochschild cohomology as the cohomology of poly-vector fields

2.1. **The Hochschild–Kostant–Rosenberg (HKR) theorem.** Let $\mathcal{L}$ be a Lie algebroid as above. The skew-symmetrization map

$$\begin{array}{rcl} \wedge^{\cdot}_{\mathcal{O}_X}\mathcal{L} & \longrightarrow & \mathcal{D}^{poly,\cdot}_{\mathcal{L},X} \\ u_1 \wedge \cdots \wedge u_m & \longmapsto & \dfrac{1}{m!}\displaystyle\sum_{\sigma\in\mathfrak{S}_m} \varepsilon^{\sigma} u_{\sigma(1)} \otimes \cdots \otimes u_{\sigma(m)} \end{array}$$

is a quasi-isomorphism of sheaves, known as the *Hochschild–Kostant–Rosenberg (or, HKR) morphism*. It induces in particular an isomorphism *of graded vector spaces*

$$\mathrm{HKR} : H^{\cdot}(X, \wedge^{\cdot}\mathcal{L}) \xrightarrow{\sim} HH^{\cdot}_{\mathcal{L}}(X)\,.$$

2.2. **A multiplicative version of the HKR morphism.** Consider the short exact sequence

$$0 \to \mathcal{L} \to U(\mathcal{L})_+^{\leq 2} \to S^2_{\mathcal{O}_X}\mathcal{L} \to 0\,,$$

where $U(\mathcal{L})_+$ denotes the augmentation ideal of $U(\mathcal{L})$, i.e. $\mathcal{L}$-differential operators vanishing on constants. This extension defines the *Atiyah class of* $\mathcal{L}$:

$$\mathrm{At}_{\mathcal{L}} \in Ext^1_X\big(S^2(T_X), T_X\big) \to Ext^1_X\big(T_X^{\otimes 2}, T_X\big) \cong$$
$$Ext^1_X\big(T_X, End(T_X)\big) \cong H^1\big(X, \Omega^1_X \otimes End(T_X)\big)\,.$$

We derive from it the *Todd genus of* $\mathcal{L}$:

$$\mathrm{Td}_{\mathcal{L}} := \det \sqrt{\frac{\mathrm{At}_{\mathcal{L}}}{1 - exp(-\mathrm{At}_{\mathcal{L}})}} \in \oplus_k H^k(X, \Omega^k_X)\,.$$

It is given by a formal expression involving sums of products of $c_k = \mathrm{tr}(\mathrm{At}^k_{\mathcal{L}})$'s.

Theorem 2.1 ([1])**.** *Composing the HKR morphism together with the contraction against the Todd genus leads to a ring isomorphism*

$$\mathrm{HKR} \circ (\mathrm{Td}_{\mathcal{L}} \llcorner -) \,:\, H^{\cdot}(X, \wedge^{\cdot}\mathcal{L}) \xrightarrow{\sim} HH^{\cdot}_{\mathcal{L}}(X)\,.$$

2.3. **Sanity check: the original HKR morphism is not multiplicative.** Let us show that when X is a K3 surface and $\mathcal{L} = T_X$ the HKR morphism is not a ring isomorphism in cohomology. Using Theorem 2.1 above this is equivalent to show that the contraction $\mathrm{Td}_{T_X} \llcorner -$ against the Todd genus is not a ring isomorphism. Note that, for degree reasons, in the case of a K3 surface the Todd genus takes the form $exp(ac_1 + bc_2)$, with a and b non-zero. Since the contraction $c_1 \llcorner -$ with c_1 is known to be a derivation, we are left to show that the contraction with c_2 is not a derivation.

Sketch of proof that $c_2 \llcorner -$ *is not a derivation.* Let ω be the symplectic form on X and Π be the corresponding Poisson bivector. Observe that c_2 is proportional to $[\omega \wedge \bar{\omega}] \in H^2(X, \Omega^2_X)$.

One the one hand, we have that $c_2 \llcorner (\Pi \wedge \Pi) = 0$ ($\Pi \wedge \Pi = 0$ because of dimension). On the other hand, $(c_2 \llcorner \Pi) \wedge \Pi = \Pi \wedge (c_2 \llcorner \Pi)$ is proportional to $[\bar{\omega} \wedge \Pi]$, which is non-zero in $H^2(X, \wedge^2 T_X)$. Hence $c_2 \llcorner -$ is not a derivation. □

References

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Hochschild cohomology of projective hypersurfaces

Liyu Liu

Hochschild cohomology originated as a cohomology theory for associative algebras, which is known to be closely related to deformation theory since the work of Gerstenhaber. Meanwhile, both the cohomology and the deformation side of the picture have been developed for a variety of mathematical objects, ranging from schemes to abelian and differential graded categories. We compute Hochschild cohomology of projective hypersurfaces $X \subset \mathbb{P}^n$ starting from the Gerstenhaber-Schack complex of the (restricted) structure sheaf. We construct a complex in terms of Čech cochains as well as an explicit quasi-isomorphism from it to the Gerstenhaber-Schack complex. The cohomology of the former is computed, and then the corresponding Gerstenhaber-Schack cocycles are derived.

We are particularly interested in the second cohomology group and its relation with deformations. We show that the group admits a decomposition

$$HH^2(X) \cong H^0(X, \wedge^2 \mathcal{T}_X) \oplus H^1(X, \mathcal{T}_X) \oplus H^2(X, \mathcal{O}_X) \oplus E$$

which is similar to the classical HKR decomposition. It is proven that a projective hypersurface is smooth if and only if the classical HKR decomposition holds for this group (i.e. $E = 0$), if and only if the classical HKR decomposition holds for all cohomology groups, namely,

$$HH^i(X) \cong \bigoplus_{p+q=i} H^p(X, \wedge^q \mathcal{T}_X)$$

for all $i \geq 0$. In most cases (for example, $n \geq 3$), a 2-class in E deforms local multiplications. In some rare cases, however, it deforms both local multiplications and restriction maps simultaneously; such a class is said to be intertwined. We catch an explicit family of intertwined 2-classes for plane singularities determined by homogeneous polynomials of degree ≥ 6, and prove the nonexistence in the case degree ≤ 4. The situation for degree five is still open.

Furthermore, we make our computations precise in the case of quartic surfaces, and obtain the lower bound of the dimensions of the second cohomology groups. Not only smooth quartic surfaces, but also some non-smooth ones, can reach the lower bound. An example of such non-smooth surfaces is a Kummer surface.

Witt vectors as a polynomial functor

Dmitry Kaledin

Recall that to any commutative ring A, one canonically associates the ring $W(A)$ of *p-typical Witt vectors* of A. Witt vectors are functorial in A, and $W(A)$ is the inverse limit of rings $W_m(A)$ of *m-truncated p-typical Witt vectors* numbered by integers $m \geq 1$. We have $W_1(A) \cong A$, and for any m, $W_{m+1}(A)$ is an extension of $W_m(A)$ by A itself.

If A is annihilated by a prime p and perfect — that is, the Frobenius endomorphism $F : A \to A$ is bijective — then one has $W_m(A) \cong W(A)/p^m$, and in

particular, $A \cong W(A)/p$. If A is not perfect, this is not true. However, if A is sufficiently nice — for example, if it is the algebra of functions on a smooth affine algebraic variety — then $W(A)$ has no p-torsion. Thus roughly speaking, the Witt vectors construction provides a functorial way to associate a ring of characteristic 0 to a ring of characteristic p.

Historically, this motivated a lot of interest in the construction. In particular, one of the earliest attempts to construct a Weil cohomology theory, due to J.-P. Serre, was to consider $H^*(X, W(\mathcal{O}_X))$, where X is an algebraic variety over a finite field k of positive characteristic p, and $W(\mathcal{O}_X)$ is the sheaf obtained by taking the Witt vectors of its structure sheaf $\mathcal{O}_X$.

This attempt did not quite work, and the focus of attention switched to other cohomology theories discovered by A. Grothendieck: étale cohomology first of all, but also cristalline cohomology introduced slightly later. Much later, P. Deligne and L. Illusie discovered what could be thought of as a vindication of Serre's original approach. They proved that any smooth algebraic variety X over a perfect field k of positive characteristic can be equipped with a functorial *de Rham-Witt complex* $W\Omega^*_X$, an extension of the usual de Rham complex Ω^*_X. In degree 0, one has $W\Omega^0_X \cong W(\mathcal{O}_X)$, but in higher degrees, one needs a new construction. The resulting complex computes cristalline cohomology $H^*_{cris}(X)$ of X, in the sense that one has a canonical isomorphism $H^*_{cris}(X) \cong H^*(X, W\Omega^*_X)$, and cristalline cohomology is known to be a Weil cohomology theory.

Yet another breakthrough in our understanding of Witt vectors happenned in 1995, and it was due to L. Hesselholt. What he did, based on ideas from algebraic topology, was to construct Witt vectors $W(A)$ for an arbitrary associative ring A. Hesselholt's $W(A)$ is also the inverse limit of its truncated version $W_m(A)$, and if A is commutative and unital, then it coincides with the classical Witt vectors ring. But if A is not commutative, $W(A)$ is not even a ring — it is only an abelian group. We have $W_1(A) = A/[A, A]$, the quotient of the algebra A by the subgroup spanned by commutators of its elements, and for any m, $W_{m+1}(A)$ is an extension of $W_m(A)$ by $A/[A, A]$.

In the context of non-commutative algebra and non-commutative algebraic geometry, one common theme of the two constructions is immediately obvious: Hochschild homology. On one hand, for any associative ring, $A/[A, A]$ is the 0-th Hochschild homology group $HH_0(A)$. On the other hand, for a smooth affine algebraic variety $X = \mathrm{Spec} A$, the spaces $H^0(X, \Omega^i_X)$ of differential forms on X are identified with the Hochschild homology groups $HH_i(A)$ by the famous theorem of Hochschild, Kostant and Rosenberg. Thus one is lead to expect that a unifying theory would use an associative unital k-algebra A as an input, and produce what one could call "Hochschild-Witt homology groups" $WHH_*(A)$ such that in degree 0, $WHH_0(A)$ coincides with Hesselholt's Witt vectors, while for a commutative A with smooth spectrum $X = \mathrm{Spec}\, A$, we would have natural identifications $WHH_i(A) \cong H^0(X, W\Omega^i_X)$.

However, Hochschild homology is in fact a theory with two variables – an algebra A and an A-bimodule M (that is, a module over the product $A^o \otimes A$ of A with

its opposite algebra A^o). To obtain $HH_*(A)$, one takes as M the diagonal bimodule A, but the groups $HH_*(A, M)$ are well-defined for any bimodule. Moreover, Hochschld homology has the following trace-like property: for any two algebras A, B, a left module M over $A^o \otimes B$, and a left module N over $B^o \otimes A$, we have a canonical isomorphism

$$HH_*(A, M \otimes_B N) \cong HH_*(B, N \otimes_A M),$$

subject to some natural compatibility conditions. It can be axiomatized under the name of "trace theory" and "trace functor", and one can prove the following: if one wants to have a generalization of Hochschild homology that is a functor of two variables A, M and has trace isomorphisms, then it suffices to define it for $A = k$. Thus one can trade the first variable for the second one: instead of constructing $WHH_*(A)$ for an arbitrary A, one can construct $WHH_*(k, M)$ for an arbitrary k-vector space M. This is hopefully simpler. In particular, it is reasonable to expect that $WHH_i(k, M) = 0$ for $i \geq 1$, so that the problem reduces to constructing a single functor from k-vector spaces to abelian groups.

In this talk, we present a very simple and direct construction of such a functor motivated by recent work of V. Vologodsky. In a nutshell, the basic idea is this: instead of trying to associate an abelian group to a k-vector space M directly, one should lift M to a free $W(k)$-module in some way, use it for the construction, and then prove that the result does not depend on the lifting. The resulting definition only works over a perfect field k of characteristic p and assumes that we already know the classical Witt vectors ring $W(k)$. However, it produces directly an inverse system of p-typical Witt vectors functors W_m, and it only uses elementary properties of cyclic groups $\mathbb{Z}/p^n\mathbb{Z}$, $n \geq 0$. The functors W_m are polynomial, thus the "polynomial functor" of the title.

On the Hochschild cohomology of finite group algebras

Markus Linckelmann

Let $\mathcal{O}$ be a complete local commutative principal ideal domain with residue field k of prime characteristic p. Let G be a finite group. A *block of* $\mathcal{O}G$ is an indecomposable direct factor B of the group algebra $\mathcal{O}G$. Then $B = \mathcal{O}G \cdot b$ for a primitive idempotent b in $Z(\mathcal{O}G)$; this correspondence is a bijection between the blocks of $\mathcal{O}G$ and the primitive idempotents in $Z(\mathcal{O}G)$. Note that a block B of $\mathcal{O}G$ is also an indecomposable direct summand of $\mathcal{O}G$ as an $\mathcal{O}G$-$\mathcal{O}G$-bimodule. Which algebras arise as block algebras of finite groups? It is expected that only 'few' algebra do arise in this way, and the finiteness conjectures in block theory formalise this intuition to some extent. In order to describe one of these conjectures, we will need the notion of a *defect group of a block*, a concept which goes back to Brauer.

Definition 1. *Let B be a block of $\mathcal{O}G$. A defect group of B is a maximal p-subgroup P of G such that $\mathcal{O}P$ is isomorphic to a direct summand of B as an $\mathcal{O}P$-$\mathcal{O}P$-bimodule.*

The defect groups of a block form a conjugacy class of p-subgroups, and there is at least one block whose defect groups are the Sylow p-subgroups of G. The blueprint for finiteness conjectures in block theory is that for a fixed finite p-group there should be only finitely many classes of blocks with a given structural property. Here is a classical example.

Conjecture 2 (Donovan's conjecture, 1970s)**.** *For any fixed finite p-group P, there are only finitely many Morita equivalence classes of block algebras of finite groups with defect groups isomorphic to P.*

Originally, this conjecture was formulated for blocks over an algebraically closed field k. The formulation over $\mathcal{O}$ makes just as much sense, although it seems unknown at present, whether the algebra structure of a block B of $\mathcal{O}G$ is determined by that of the corresponding block $k \otimes_{\mathcal{O}} B$ of kG.

Donovan's conjecture is known to hold in a number of cases, including P cyclic (Janusz, Kupisch, 1970s), most cases of 2-groups of tame representation type (Erdmann, 1980s), and for all elementary abelian 2-groups (by recent work of Eaton, Kessar, Külshammer, and Sambale, using the classification of finite simple groups). It is possible to reformulate Donovan's conjecture without reference to defect groups, using the following notion.

Definition 3 ([5, 3.1])**.** *Two $\mathcal{O}$-algebras A and B are called separably equivalent if there exist an A-B-bimodule M and a B-A-bimodule N, both finitely generated projective as left and right modules, such that A is isomorphic to a direct summand of $M \otimes_B N$ as an A-A-bimodule, and B is isomorphic to a direct summand of $N \otimes_A M$ as a B-B-bimodule.*

A block algebra B of $\mathcal{O}G$ and any of its defect group algebras $\mathcal{O}P$ are separably equivalent, via the bimodules $B_{\mathcal{O}P}$ and ${}_{\mathcal{O}P}B$, obtained from restricting the B-B-bimodule B to $\mathcal{O}P$ on one side. Further examples of separable equivalences between $\mathcal{O}$-algebras include Morita equivalences, stable equivalences of Morita type (Broué, 1990s) and singular equivalences of Morita type (Chen-Sun, 2012). In order to reformulate Dononvan's conjecture without referring to defect groups, we need the following observation.

Proposition 4. *Let P and Q be finite p-groups. Suppose that $\mathcal{O}$ has characteristic zero and that $\mathcal{O}P$ and $\mathcal{O}Q$ are separably equivalent. Then the following hold.*
(i) $|P| = |Q|$.
(ii) $\mathrm{rk}(P) = \mathrm{rk}(Q)$.

Corollary 5. *Donovan's conjecture (over $\mathcal{O}$ having characteristic zero, with k algebraically closed) is equivalent to the following conjecture: every separable equivalence class of block algebras of finite groups consists of finitely many Morita equivalence classes.*

Question 6. *Which algebras (other than block algebras) have the property that their separable equivalence class consists of finitely many Morita equivalence classes?*

Following the terminology introduced in [3], we say that a finite-dimensional k-algebra A satisfies the condition (Fg) if $HH^*(A)$ is Noetherian and $\mathrm{Ext}^*_A(U,U)$ is finitely generated as a module over $HH^*(A)$, for any finitely generated A-module U. As a consequence of a theorem of Evens and Venkov, finite group algebras satisfy the condition (Fg).

An $\mathcal{O}$-algebra A is called *symmetric* if A is finitely generated as an $\mathcal{O}$-module and if A is isomorphic, as an A-A-bimodule, to its $\mathcal{O}$-dual $A^\vee = \mathrm{Hom}_{\mathcal{O}}(A,\mathcal{O})$. Examples of symmetric $\mathcal{O}$-algebras include finite group algebras, Iwahori-Hecke algebras of finite Coxeter groups, and Hopf algebras with an antipode of order 2.

Theorem 7 ([5, 4.1])**.** *Let A and B be separably equivalent symmetric k-algebras. Then A satisfies (Fg) if and only if B satisfies (Fg). Moreover, in that case, $HH^*(A)$ and $HH^*(B)$ have the same Krull dimension.*

Theorem 8 ([5, 1.1])**.** *Let $\mathcal{H} = \mathcal{H}_q(W,S)$ be an Iwahori-Hecke algebra over $\mathbb{C}$ of a finite Coxeter group (W,S), with parameter $q \in \mathbb{C}^\times$ of finite order. Suppose that all irreducible components of W are of classical type A, B, or D, and that if W involves a component of type B or D, then q has odd order. Then $\mathcal{H}$ satisfies the condition (Fg).*

It is not known in general, whether Iwahori-Hecke algebras of finite Coxeter groups over fields of positive characteristic satify the condition (Fg). Donovan's conjecture would in particular imply that there are only finitely many Hochschild cohomology algebras, up to isomorphism, of blocks with a fixed defect group. While in this generality still unknown, this can be shown to become true upon passing to Hilbert series.

Theorem 9 ([4, Theorem 2])**.** *Let P be a finite p-group. There are only finitely many power series in $\mathbb{Z}[[t]]$ which are equal to $\sum_{n\geq 0} \dim_k(HH^n(B)) \cdot t^n$, with B running over the blocks of finite group algebras over k with defect groups isomorphic to P.*

The proof uses a result of Symonds, previously conjectured by Benson, stating that the Castelnuovo-Mumford regularity of finite group cohomology is zero.

Separable equivalences preserve further the representation type as well as the dimensions of stable and derived module categories as triangulated categories; see [5]. Separable equivalences preserve in some cases the finitisic dimension of endomorphism algebras of modules; this has been used in [6] to determine the finitisic dimension of the category of cohomological Mackey functors of a block, extending work of Tambara.

Question 10. *Do separable equivalences between symmetric k-algebras preserve in general the Castelnuovo-Mumford regularity of Hochschild cohomology?*

We assume now that $\mathcal{O}$ has characteristic zero, and we denote by K the field of factions of $\mathcal{O}$.

Definition 11. *Let A be a symmetric $\mathcal{O}$-algebra and U an A-lattice (that is, U is a finitely generated A-module which is free as an $\mathcal{O}$-module). For $\alpha \in \mathrm{End}_{\mathcal{O}}(U)$, denote by $\mathrm{tr}_U(\alpha)$ the trace of α on U.*

(a) *U is called a Knörr lattice, or U is said to have the property (K), if for any $\alpha \in \mathrm{End}_A(U)$, we have $\mathrm{tr}_U(\alpha) \cdot \mathcal{O} \subseteq \mathrm{rk}(U) \cdot \mathcal{O}$, with equality if and only if α is an automorphism of U.*

(b) *U is said to have the stable exponent property or, for short, the property (E), if the socle of the $\underline{\mathrm{End}}_A(U)$ as a module over itself is equal to $\lambda \underline{\mathrm{Id}}_U \cdot \mathcal{O}$ for some $\lambda \in \mathcal{O}$.*

Theorem 12 (Carlson-Jones [1])**.** *Let G be a finite group and U an indecomposable non-projective $\mathcal{O}G$-lattice. Then U has property (K) if and only if U has property (E).*

An intriguing aspect of this theorem is that property (E) is invariant under stable equivalences, but property (K) is not even invariant under Morita equivalences. And yet, these two properties are equivalent for finite group algebras. Thus a finite group algebra $\mathcal{O}G$ must have a property which singles it out within its Morita equivalence class.

What follows is based on joint work with F. Eisele, M. Geline, and R. Kessar [2], where we identify such a property in terms of Tate duality. For A a symmetric $\mathcal{O}$-algebra and any two A-lattices U, V, Tate duality is a natural pairing

$$\underline{\mathrm{Hom}}_A(U,V) \times \underline{\mathrm{Hom}}_A(V,U) \to K/\mathcal{O}$$

In the case of finite group algebras, it is possible to be more precise: one can replace $K/\mathcal{O}$, which is the colimit of all torsion $\mathcal{O}$-modules of the form $\mathcal{O}/J(\mathcal{O})^n$, by the module $\mathcal{O}/|G|\mathcal{O}$, and the duality sends $(\underline{\alpha}, \underline{\beta}) \in \underline{\mathrm{Hom}}_A(U,V) \times \underline{\mathrm{Hom}}_A(V,U)$ to the image in $\mathcal{O}/|G|\mathcal{O}$ of $\mathrm{tr}_U(\beta \circ \alpha)$, where α, β are representatives of $\underline{\alpha}$, $\underline{\beta}$, respectively. This is a key ingredient in the proof of the above theorem of Carlson and Jones, since this is where stable endomorphisms and traces are being connected. It turns out that the property of $\mathcal{O}G$ which yields this description of Tate duality can be identified as follows. Let A be a symmetric $\mathcal{O}$-algebra; choose a bimodule isomorphism $A \cong A^\vee$. Dualising the multiplication map $\mu : A \otimes_{\mathcal{O}} A \to A$ yields a bimodule homomorphism $A^\vee \to (A \otimes_{\mathcal{O}} A)^\vee$. Using the isomorphism $A \cong A^\vee$, this yields a bimodule homomorphism $\tau : A \to A \otimes_{\mathcal{O}} A$. Then $\mu \circ \tau$ is a bimodule endomorphism of A, hence sends 1_A to an element $z_A \in Z(A)$. We call z_A the *relative projective element of A*; this is also called the *central Casimir element in A*. The element z_A depends on the choice of the bimodule isomorphism $A \cong A^\vee$; that is, z_A is unique up to multiplication by elements in $Z(A)^\times$.

Definition 13. *A symmetric $\mathcal{O}$-algebra A is said to have the projective scalar property, if there is a choice of a bimodule isomorphism $A \cong A^\vee$ such that $z_A = \lambda \cdot 1_A$ for some $\lambda \in \mathcal{O}$.*

Remarks 14.

(a) *Finite group algebras, block algebras, their source algebras, and matrix algebras have the projective scalar property.*

(b) *The projective scalar property is not invariant under Morita equivalences in general.*

(c) *There are examples of Morita equivalence classes of symmetric $\mathcal{O}$-algebras having no representative satisfying the projective scalar property, so this property can be used in principle to rule out certain Morita equivalence classes of $\mathcal{O}$-algebras as coming from block algebras.*

(d) *Iwahori-Hecke algebras may or may not have the projective scalar property; this depends on the underlying parameter q.*

(e) *Hopf algebras with an antipode of order 2 have the projective scalar property.*

Theorem 15 ([2])**.** *Let A be a symmetric $\mathcal{O}$-algebra such that for some bimodule isomorphism $A \cong A^\vee$ we have $z_A = \lambda \cdot 1_A$, for some $\lambda \in \mathcal{O}$. Suppose that $K \otimes_{\mathcal{O}} A$ is split semisimple. Then Tate duality takes the form*

$$\underline{\mathrm{Hom}}_A(U,V) \times \underline{\mathrm{Hom}}_A(V,U) \to \mathcal{O}/\lambda\mathcal{O}$$

mapping a pair $(\underline{\alpha}, \underline{\beta}) \in \underline{\mathrm{Hom}}_A(U,V) \times \underline{\mathrm{Hom}}_A(V,U)$ to the image in $\mathcal{O}/\lambda\mathcal{O}$ of $\mathrm{tr}_U(\beta \circ \alpha)$.

And whenever Tate duality takes this form, the proof of the above theorem of Carlson and Jones can be adapted to showing the following result.

Theorem 16 ([2])**.** *Let A be a symmetric $\mathcal{O}$-algebra which has the projective scalar property. Suppose that $K \otimes_{\mathcal{O}} A$ is split semisimple. An indecomposable non-projective A-lattice U has the property (K) if and only if it has the property (E).*

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Towards coherent duality over derived formal schemes

Liran Shaul

Given a finite type map $f : X \to Y$ between noetherian schemes, coherent duality theory focuses on a pseudofunctor $f^! : D^+_{\mathrm{c}}(Y) \to D^+_{\mathrm{c}}(X)$, which, when f is proper, is right adjoint to the derived pushforward pseudofunctor $\mathrm{R}f_*$. One important property of $f^!$ is its base change property: given a quasi-compact quasi-separated morphism of schemes $f : X \to S$, and given any morphism $g : Y \to S$, let

$X' := X \times_S Y$, with projections $f' : X' \to Y$ and $h : X' \to X$; if X and Y are tor-independent over S, then there is an isomorphism of functors $\mathrm{R}f'_*\mathrm{L}h^*(-) \cong \mathrm{L}g^*\mathrm{R}f_*(-)$. If one wishes to understand how $(-)^!$ behaves with respect to more general base changes, one needs to replace the fiber product with its (non-linear) derived functor $X \times^{\mathrm{R}}_S Y$, which is in general no longer an ordinary scheme, but a derived scheme.

In this talk, which is based on [5], we explain how to construct $(-)^!$ over a suitable category of affine derived schemes, and show that this extended $(-)^!$ commutes with base change with respect to maps which are of finite flat dimension. Our construction is based on the notion of a rigid dualizing complex. Rigid dualizing complexes, first introduced by Van den Bergh in [6], are dualizing complexes that carry an extra structure, making them unique up to a unique isomorphism.

Recall that if A is a noetherian derived ring, by [2, 3, 8], a complex $R \in \mathrm{D}^{\mathrm{b}}_{\mathrm{f}}(A)$ is called a dualizing complex if it has finite injective dimension over A, and the natural map $A \to \mathrm{R}\operatorname{Hom}_A(R, R)$ is an isomorphism in $\mathrm{D}(A)$. Given a base ring K, and a derived algebra A over K, a rigid dualizing complex over A relative to K is a dualizing complex over A, of finite flat dimension over K, and a specified isomorphism

$$\rho : R \to \mathrm{R}\operatorname{Hom}_{A \otimes^{\mathrm{L}}_K A}(A, R \otimes^{\mathrm{L}}_K R).$$

The right hand side, a variant of derived Hochschild cohomology, is well defined by [7, Theorem 0.3.4] (or, if A is a ring, by [1, Theorem 3.2]).

Given a noetherian ring K, we denote by $\mathrm{DGR}_{ef.fd/K}$ the category of noetherian derived rings A over K, such that $\mathrm{H}^0(A)$ is essentially of finite type over K, and such that A has finite flat dimension over K. Our first main result concerns existence and uniqueness of rigid dualizing complexes in this category:

Theorem 1. *Let K be a Gorenstein noetherian ring of finite Krull dimension. For any $A \in \mathrm{DGR}_{ef.fd/K}$, there exists a rigid dualizing complex R_A over A relative to K, and moreover, it is unique up to isomorphism in $\mathrm{D}(A)$.*

Next, we discuss functoriality. A map $f : A \to B$ between noetherian derived rings is called cohomologically finite if the induced map $\mathrm{H}^0(f) : \mathrm{H}^0(A) \to \mathrm{H}^0(B)$ is a finite ring map.

Theorem 2. *Let K be a Gorenstein noetherian ring of finite Krull dimension, and let $f : A \to B$ be a cohomologically finite map in $\mathrm{DGR}_{ef.fd/K}$. If R_A is the rigid dualizing complex over A relative to K, then $\mathrm{R}\operatorname{Hom}_A(B, R_A)$ has the structure of a rigid dualizing complex over B relative to K.*

A map $f : A \to B$ between noetherian derived rings is called cohomologically essentially smooth if B has flat dimension 0 over A, and the induced map $\mathrm{H}^0(f) : \mathrm{H}^0(A) \to \mathrm{H}^0(B)$ is essentially smooth (i.e, it is essentially of finite type and formally smooth). In this case, $\Omega^1_{\mathrm{H}^0(B)/\mathrm{H}^0(A)}$ is locally free, and

$$\omega_{\mathrm{H}^0(B)/\mathrm{H}^0(A)} := (\mathrm{H}^0(f))^!(\mathrm{H}^0(A))$$

is a tilting complex over $H^0(B)$ (locally, on each connected component of $\operatorname{Spec}(\mathrm{H}^0(B))$, it is given by $\Omega^n_{\mathrm{H}^0(B)/\mathrm{H}^0(A)}[n]$). It follows that there is a unique

$\omega_{B/A} \in \mathrm{D}(B)$ such that

$$\omega_{B/A} \otimes^{\mathrm{L}}_{B} \mathrm{H}^0(B) \cong \omega_{\mathrm{H}^0(B)/\mathrm{H}^0(A)}$$

Theorem 3. *Let K be a Gorenstein noetherian ring of finite Krull dimension, and let $f : A \to B$ be a cohomologically essentially smooth map in $\mathrm{DGR}_{ef.fd/K}$. If R_A is the rigid dualizing complex over A relative to K, then $R_A \otimes^{\mathrm{L}}_{A} \omega_{B/A}$ has the structure of a rigid dualizing complex over B relative to K.*

Using these results, we obtain a $(-)^!$ theory over $\mathrm{DGR}_{ef.fd/K}$.

Theorem 4. *Let K be a Gorenstein noetherian ring of finite Krull dimension. There exists a pseudofunctor*

$$(-)^! : \mathrm{DGR}_{ef.fd/K} \to \mathbf{Cat}$$

with the following properties:

1. *On the full subcategory of $\mathrm{DGR}_{ef.fd/K}$ made of essentially finite type K-algebras which are of finite flat dimension over K, $(-)^!$ is naturally isomorphic to the classical twisted inverse image pseudofunctor.*
2. *Given a cohomologically finite map $f : A \to B$ in $\mathrm{DGR}_{ef.fd/K}$, there is an isomorphism*
$$f^!(M) \cong \mathrm{R}\,\mathrm{Hom}_A(B, M)$$
of functors $\mathrm{D}^+_{\mathrm{f}}(A) \to \mathrm{D}^+_{\mathrm{f}}(B)$.
3. *Given a cohomologically essentially smooth map $f : A \to B$ in $\mathrm{DGR}_{ef.fd/K}$, there is an isomorphism*
$$f^!(M) \cong M \otimes^{\mathrm{L}}_{A} \omega_{B/A}$$
of functors $\mathrm{D}^+_{\mathrm{f}}(A) \to \mathrm{D}^+_{\mathrm{f}}(B)$.

Returning to our original motivation, and, by taking K-flat resolutions, we obtain the following non-flat base change result:

Theorem 5. *Let K be a Gorenstein noetherian ring of finite Krull dimension, let $f : A \to B$ be an arbitrary map in $\mathrm{DGR}_{ef.fd/K}$, and let $g : A \to C$ be a K-flat map in $\mathrm{DGR}_{ef.fd/K}$ such that C has finite flat dimension over A. Consider the induced base change commutative diagram*

$$\begin{array}{ccc} A & \xrightarrow{\quad f \quad} & B \\ \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle h} \\ C & \xrightarrow{\quad f' \quad} & B \otimes_A C \end{array}$$

Then there is an isomorphism

$$\mathrm{L}h^* \circ f^!(-) \cong (f')^! \circ \mathrm{L}g^*(-)$$

of functors

$$\mathrm{D}^+_{\mathrm{f}}(A) \to \mathrm{D}^+_{\mathrm{f}}(B \otimes_A C).$$

This, based on [5], concludes the development of an affine derived theory of $(-)^!$. Going further, we would like to obtain a similar theory for affine derived formal schemes. As the above theory is based on derived Hochschild cohomology, it becomes necessary to make a detailed study of the derived Hochschild cohomology of derived adic rings. A first step in this direction was done in our [4], where we explored relations between derived completion and derived Hochschild cohomology. To go further, one must improve the understanding of the derived torsion and derived completion functors over commutative derived rings.

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Loop homology and Hochschild cohomology

Don Stanley

Let M be a closed simply connected manifold of dimension n. The cochain functor with some field coefficients is denoted by $C^*(\_)$. The free loops space on M is denoted by LM. The loop homology $\mathbb{H}_*(LM)$ is defined to be a shift of the normal homology, $\mathbb{H}_*(LM) = H_{*+n}(LM)$.

The main focus of the talk is to look at the isomorphism

$$HH^*(C^*(M), C^*(M)) \cong \mathbb{H}_*(LM)$$

We look at this isomorphism with four levels of structure.

1) Linear
2) As algebras
3) As Gerstenhaber algebras
4) As BV algebras.

We consider the first three structures on the Hochschild cohomology to be well known, but say a bit about the fourth structure. For the loop homology we describe the algebra, Gerstenhaber and BV algebra structures.

Given a base point $*$ in a topological space X, there is the familar (based) loops base ΩX. This is the set of base point preserving maps, from the circle S^1 (considered as the complex numbers of length 1) into X. These maps can be considered as paths beginning and ending at the base point. This is topologized with the compact open topology.

$$\Omega X = \{\phi : S^1 \to X | \phi(1) = *\}$$

As the paths begin and end at the base point, we can compose them to get a multiplication on ΩX. In turn this makes $H_*(\Omega X)$ into an associative algebra.

The free loop space on X, LX consists of all maps from S^1 to X, again with the compact open topology.

$$LX = \{\phi \colon S^1 \to X\}$$

Even though we can only compose loops that have the same base point, when $X = M$, Chas and Sullivan constructed a so called loop product on $H_*(LM)$, by mixing the intersection product on $H_*(M)$ and the product on ΩM mentioned above.

Consider the fibration

$$\Omega M \to LM \stackrel{ev_1}{\to} M$$

where ev_1 is evaluation at the base point of S^1. Two free loops in LM can be composed if they evaluate to the same point in M. So if $\alpha \in C_l(LM)$ and $\beta \in C_k(LM)$ intersect nicely we can get a chain $\alpha \circ \beta \in C_{l+k-n}(LM)$, by only taking the points in the simplices where the evaluations of the chains are the same. This induces a graded algebra structure on $\mathbb{H}_*(LM)$.

It turns out that this algebra structure is commutative and the homotopy making it commute gives rise to a Lie algebra structure (of degree 1) on $\mathbb{H}_*(LM)$ and turns $\mathbb{H}_*(LM)$ into a Gerstenhaber algebra.

Chas and Sullivan also describe an operator $B \colon \mathbb{H}_*(LM) \to \mathbb{H}_{*+1}(LM)$ which is simply induced by the circle action $S^1 \times LM \to LM$. You just take the image of multiplication by the fundamental class of the circle. This operator turns $\mathbb{H}_*(LM)$ into a BV algebra which induces the Gerstenhaber algebra structure.

On the $HH^*(C^*(M), C^*(M))$ side the BV structure comes from Connes cyclic operator on HH_* together with the duality equivalence $C^*(M) \to C_*(M)$. The structure required of the equivalence is somewhat complicated in general but simplifies over the rationals when it can be replaced by an isomorphism.

With general coefficients the isomorphism $HH^*(C^*(M), C^*(M)) \cong \mathbb{H}_*(LM)$ linearly and as algebras has been proven by many authors. As Gerstenhaber and BV algebras it has been proven over the rationals, however over the integers or finite characteristic fields it has been shown that the two sides are not in general isomorphic as BV algebras.

Tensor products with Carlson's L_ζ-modules

Srikanth B. Iyengar
(joint work with Jon F. Carlson)

Let k be a field and A a k-algebra. In what follows $\otimes$ denotes tensor products over k, that is to say, $\otimes_k$. We say that a map $\Delta\colon A \to A \otimes A$ is a *coproduct* on A if the k-algebra A can be endowed with a structure of a Hopf algebra with diagonal Δ. It can happen that A has different coproducts.

Example 1. Let k be a field of positive characteristic p and set

$$A := k[x_1, \dots, x_c]/(x_1^p, \dots, x_c^p)\,,$$

an algebra of truncated polynomials.

One can view A as the group algebra of the elementary abelian p-group $(\mathbb{Z}/p\mathbb{Z})^c$ with generators $\{x_i + 1\}_{i=1}^c$. Then A has the coproduct defined by

$$\Delta_{\mathrm{Gr}}(x_i) := x_i \otimes 1 + x_i \otimes x_i + 1 \otimes x_i\,.$$

One can also view A as the restricted enveloping algebra of the abelian Lie algebra with basis $\{x_i\}_{i=1}^c$. Then A has the coproduct defined by

$$\Delta_{\mathrm{Lie}}(x_i) := x_i \otimes 1 + 1 \otimes x_i\,.$$

These will be the running examples in this text.

Let Δ be a coproduct on A and let M, N be (left) A-modules. Then $M \otimes N$ is an $A \otimes A$-module; we write $\Delta^*(M \otimes N)$ for A-module obtained by restriction of scalars along Δ. Thus, for any α in A and $m \otimes n$ in $M \otimes N$, one has

$$\alpha(m \otimes n) := \sum_{(\alpha)} \alpha_1 m \otimes \alpha_2 n \quad \text{where } \Delta(\alpha) = \sum_{(\alpha)} \alpha_1 \otimes \alpha_2.$$

The question that informs this work is this: *How does this A-module depend on Δ?* Certainly, as Δ changes, one can get non-isomorphic A-modules; see Example 4. On the other hand, for any Δ one has

$$\Delta^*(M \otimes k) \cong M \quad \text{and} \quad \Delta^*(M \otimes A) \cong A^r \quad \text{where } r = \operatorname{rank}_k M.$$

Carlson modules. Fix a class ζ in $\operatorname{Ext}_A^d(k,k)$. This is represented by a homomorphism $\Omega^d k \to k$, where $\Omega^d k$ denotes a d-th syzygy module of k. We can assume that this map is surjective; its kernel is the module that is usually denoted L_ζ:

$$0 \longrightarrow L_\zeta \longrightarrow \Omega^d k \longrightarrow k \longrightarrow 0\,. \tag{1}$$

This module was introduced by Carlson [1, pp. 293], in the context of group algebras, and plays an important role in the theory of support varieties. We are interested in the modules $\Delta^*(L_\zeta \otimes M)$, where Δ is a coproduct on A.

Example 2. Let A be the k-algebra introduced in Example 1. Then there are isomorphisms of k-algebras

$$\operatorname{Ext}_A(k,k) \cong \begin{cases} k[\eta_1, \dots, \eta_c] & \text{when } p = 2 \\ \Lambda_k(\eta_1, \dots, \eta_c) \otimes k[\zeta_1, \dots, \zeta_c] & \text{when } p \geq 3 \end{cases}$$

Here each η_i has degree one and ζ_i has degree 2; this computation can be found in, for example, [3, Corollary 3.5.7]. We are interested in the following subalgebra.

$$S := \begin{cases} k[\eta_1^2, \dots, \eta_c^2] & \text{when } p = 2 \\ k[\zeta_1, \dots, \zeta_c] & \text{when } p \geq 3 \end{cases}$$

The result below on the coproducts introduced in Example 1 is from [2].

Theorem 3. *With notation as above, if ζ is in S, then the A-modules $\Delta^*_{\mathrm{Gr}}(L_\zeta \otimes M)$ and $\Delta^*_{\mathrm{Lie}}(L_\zeta \otimes N)$ are isomorphic for any finitely generated A-module M.*

Such an isomorphism need not hold if ζ is not in S.

Example 4. Let k be a field of characteristic 2 and set $A = k[x_1, x_2]/(x_1^2, x_2^2)$. With notation as in Example 2, set $\zeta = \eta_1 + c\eta_2$, where $c \in k \setminus \{0, 1\}$. Then $\Delta^*_{\mathrm{Gr}}(L_\zeta \otimes L_\zeta)$ is indecomposable, whereas $\Delta^*_{\mathrm{Lie}}(L_\zeta \otimes L_\zeta)$ decomposes as $L_\zeta \oplus L_\zeta$.

We now outline a proof of Theorem 3. For a start the exact sequence (1) induces an exact sequence of A-modules

$$0 \longrightarrow \Delta^*(L_\zeta \otimes M) \longrightarrow \Delta^*(\Omega^d k \otimes M) \longrightarrow \Delta^*(k \otimes M) \longrightarrow 0\,.$$

Note that $\Delta^*(\Omega^d k \otimes M)$ is a d-th syzygy module of M; this identification does depend on Δ. One thus gets an exact sequence of A-modules

$$0 \longrightarrow \Delta^*(L_\zeta \otimes M) \longrightarrow \Omega^d k \xrightarrow{\Delta^*(\zeta \otimes M)} k \longrightarrow 0$$

where $\Delta^*(\zeta \otimes M)$ denotes the image of ζ under the map of k-algebras

$$(-) \otimes M \colon \operatorname{Ext}^*_A(k, k) \longrightarrow \operatorname{Ext}^*_A(M, M)\,.$$

The result below, to be used in the proof of the theorem above, is easy to verify.

Lemma 5. *Let Δ_1 and Δ_2 be coproducts on A, and ζ an element in $\operatorname{Ext}^d_A(k, k)$. If $\Delta_1^*(\zeta \otimes M) = \Delta_2^*(\zeta \otimes M)$, then up to projective summands, the A-modules $\Delta_1^*(L_\zeta \otimes M)$ and $\Delta_2^*(L_\zeta \otimes M)$ are isomorphic.* □

Hochschild cohomology. Fix a coproduct Δ on A. For any A-module X the left A-module $\Delta^*(X \otimes A)$ has also a right A-module structure, inherited by the right action on A. It is thus a module over A^{e}, the enveloping algebra of A, with

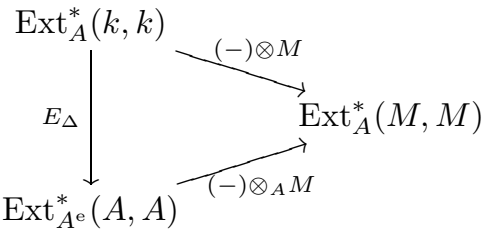
$$(\alpha \otimes \beta) \cdot (x \otimes a) = \sum_{(\alpha)} \alpha_1 x \otimes \alpha_2 a \beta$$

Write $F_\Delta(X)$ for this A^{e}-module. This construction has the following properties:

(1) $F_\Delta(A) \cong A^{\mathrm{e}}$.
(2) $F_\Delta(k) \cong A$, where A has the canonical A^{e}-module structure.
(3) $F_\Delta(X) \otimes_A M \cong \Delta^*(X \otimes M)$, as A-modules.

These assertions can be verified using the fact that Δ is coassociative and that the antipode is the inverse of the identity under the convolution product. From these one gets the following result of Pevtsova and Witherspoon [5, Lemma 13].

Proposition 6. *The following diagram of k-algebra is commutative:*



where E_Δ is induced by the functor $F_\Delta\colon \operatorname{Mod} A \to \operatorname{Mod} A^{\mathrm{e}}$ described above. □

From this result and Lemma 5 one immediately gets the following.

Corollary 7. *Give coproducts Δ_1, Δ_2 on A and an element ζ in $\mathrm{Ext}^d_A(k,k)$, if $E_{\Delta_1}(\zeta) = E_{\Delta_2}(\zeta)$, then for any A-module M, the A-modules $\Delta_1^*(L_\zeta \otimes M)$ and $\Delta_2^*(L_\zeta \otimes M)$ are isomorphic up to projective summands.* □

Example 8. Let A be as in Example 1. Then there are isomorphisms of k-algebras

$$\mathrm{Ext}_{A^{\mathrm{e}}}(k,k) \cong \begin{cases} A[\eta_1,\dots,\eta_c] & \text{when } p=2 \\ \Lambda_A(\eta_1,\dots,\eta_c)\otimes_A A[\zeta_1,\dots,\zeta_c] & \text{when } p\geq 3 \end{cases}$$

Here each η_i has degree one and each ζ_i has degree 2; this follows from Example 2 and [4, Theorem 2.1]. A direct computation then yields the following:

$$\begin{aligned} E_{\Delta_{\mathrm{Gr}}}(\eta_i) &= (1+x_i)\eta_i \quad \text{and} \quad E_{\Delta_{\mathrm{Gr}}}(\zeta_i) = \zeta_i \\ E_{\Delta_{\mathrm{Lie}}}(\eta_i) &= \eta_i \quad\quad\quad\;\; \text{and} \quad E_{\Delta_{\mathrm{Lie}}}(\zeta_i) = \zeta_i \end{aligned}$$

Given these computations the result below is evident.

Corollary 9. *$E_{\Delta_{\mathrm{Gr}}} = E_{\Delta_{\mathrm{Lie}}}$ on the subalgebra S in Example 2.* □

Combining this with Corollary 7 essentially proves Theorem 3; one only has to verify that any finite dimensional A-modules M and N of the same rank (over k) that are isomorphic up to projective summands are in fact isomorphic. This is because any projective A-module is free.

Theorem 3 has applications to the study of modular representations of elementary abelian groups; in particular, to constructing modules with prescribed annihilators in cohomology. The reader is invited to [2] for details.

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Auslander Theorem and Searching for Noncommutative McKay

James Zhang

A famous theorem of Auslander states that, if G is a finite small subgroup of $\mathrm{GL}_n(\mathbb{C})$ acting on $R := \mathbb{C}[x_1, \cdots, x_n]$ naturally, then there is a natural isomorphism of algebras

$$R * G \cong \mathrm{End}_{R^G}(R),$$

see [A1, A2]. This result plays an important role in the classical McKay correspondence. In order to extend the McKay correspodence to the noncommutative setting, one needs to establish an Auslander theorem for noncommutative algebras. By replacing "small group" by "homologically small Hopf action" we are able to prove the following.

Theorem: Let H be a semisimple Hopf algebra acting on a noetherian Artin-Schelter regular and Cohen-Macaulay algebra R homogeneously. Then the following are equivalent.

(1) The H-action on R is homologically small.
(2) There is a natural isomorphism of algebras $R\#H \cong \mathrm{End}_{R^H}(R)$.

This talk is based on joint work with Y.-H. Bao and J.-W. He [B1, B2].

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Degeneration in triangulated categories

Alexander Zimmermann
(joint work with Bernt Tore Jensen, Xiuping Su; Manuel Saorín)

1. The classical situation

Let k be an algebraically closed field and let A be a finite dimensional k-algebra. Then an A-module structure on k^d is just a k-algebra homomorphism $A \to Mat_{d\times d}(k)$. Two such maps give isomorphic module structures if and only if they are conjugate by a matrix in $GL_d(k)$. Hence the set of A-module structures on k^d form an algebraic affine variety $mod(A, d)$ on which $GL_d(k)$ acts, and orbits of this action correspond to isomorphism classes. The module M degenerates to N (denoted $M \leq_{\mathrm{deg}} N$) if N belongs to the Zariski closure of the orbit of M. How to characterise this algebraically? This is solved in the following result.

Theorem 1. *(Riedtmann* [3]*, Zwara* [10]*) Let k be an algebraically closed field and let A be a finite dimensional k-algebra, then for any two A-modules M and N we get $M \leq_{deg} N$ if and only if there is an A-module Z and a short exact sequence $0 \to Z \to Z \oplus M \to N \to 0$. We denote this second condition by $M \leq_{Zwara} N$.*

The geometric version $\leq_{\mathrm{deg}}$ is a partial order on the set of isomorphism classes of finite dimensional A-modules, as is easily seen.

2. Carrying the algebraic degeneration to the triangulated world

The goal of our research is to carry these constructions to the setting of triangulated categories. The first easy step is to generalise $\leq_{\mathrm{Zwara}}$ to triangulated categories. k denotes from now on a commutative ring, and occasionally a field.

Definition 2. *(Jensen-Su-Zimmermann* [1, 2]*, Yoshino* [8]*) Let $\mathcal{T}$ be a triangulated category. Then for any two objects M and N we denote $M \leq_{\Delta} N$ if there is an object Z and a distinguished triangle $Z \to M \oplus Z \to N \to Z[1]$.*

In [2] we proved partial order properties of $\leq_{\Delta}$. In particular we show

Theorem 3. [2] *Let $\mathcal{T}$ be a k-linear triangulated category with split idempotents.*

- *If all endomorphism algebras of objects of $\mathcal{T}$ are artinian, then $\leq_{\Delta}$ is reflexive and transitive on isomorphism classes of objects of $\mathcal{T}$.*
- *If $Hom_{\mathcal{T}}(X, Y)$ is of finite k-length for all objects X and Y, and if there is $n \in \mathbb{Z} \setminus \{0\}$ such that $Hom_{\mathcal{T}}(M, N[n]) = 0$, then*

$$M \leq_{\Delta} N \leq_{\Delta} M \Rightarrow M \simeq N.$$

We see that the pre-order property is relatively general, whereas the partial order property needs strong hypotheses. We should mention that Peter Webb showed in [7] the antisymmetry by completely different methods for triangulated categories which admit almost split triangles. Singular categories are one of our main intended application, and there the hypotheses are not satisfied. Zhengfang Wang proved in [6] independently that $\leq_{\Delta}$ is a partial order on isomorphism classes of the singular category $D_{sg}(A)$ of a finite dimensional algebra A.

Remark 4. *We should mention that this concept may allow to compare modules of different dimension. Indeed, it may happen that M is an indecomposable module of infinite projective dimension and $M \leq_{Zwara} N_1 \oplus N_2$ where N_1 is a non-zero module of finite projective dimension, and N_2 is a module of infinite projective dimension. Then $M \leq_{\Delta} N_2$ in $D_{sg}(A)$. A more explicit example is the following: If A is a self-injective algebra, and P is an indecomposable projective A-module with non zero submodule S, then $P \leq_{deg} S \oplus P/S$ and therefore $P \leq_{\Delta+nil} S \oplus P/S$ in the stable category of A-modules. But $P \simeq 0$ in the stable category, whereas the pieces S and P/S are not. Nevertheless, by definition, degeneration preserves the class in the Grothendieck group.*

3. Geometric degeneration in triangulated categories

In joint work with Saorín we concentrated on a geometric definition for degeneration in triangulated categories. We model our geometric version on Yoshino's concept of a degeneration along a dvr.

Definition 5. *(Yoshino) Let k be a field and let A be a k-algebra. Then for any two A-modules M and N we say that $M \leq_{dvr} N$ if there is a discrete valuation k-algebra V with maximal ideal tV and $k = V/tV$ and a V-flat $V \otimes_k A$-module Q such that $Q/tQ \simeq N$ as A-modules, and $Q[\frac{1}{t}] \simeq M \otimes_k V[\frac{1}{t}]$.*

For a triangulated category $\mathcal{C}_V^\circ$ and an element $t : id_{\mathcal{C}_V^\circ} \to id_{\mathcal{C}_V^\circ}$ (i.e. an element t in the centre of $\mathcal{C}_V^\circ$) we can form the Gabriel-Zisman localisation $\mathcal{C}_V^\circ \xrightarrow{p} \mathcal{C}_V^\circ[t^{-1}]$, which is again triangulated and is universal amongst all triangulated categories in which t_X becomes invertible for all objects X.

Definition 6. [4] *Let $\mathcal{C}_k^\circ$ be a triangulated k-category with split idempotents.*

- *A degeneration data for $\mathcal{C}_k^\circ$ is given by triangulated k-categories $\mathcal{C}_V^\circ$, $\mathcal{C}_k$ and $\mathcal{C}_V$ with split idempotents, such that $\mathcal{C}_V^\circ$ is full triangulated subcategory of $\mathcal{C}_V$, and $\mathcal{C}_k^\circ$ is full triangulated subcategory of $\mathcal{C}_k$, a triangle functor $\uparrow_k^V : \mathcal{C}_k \to \mathcal{C}_V$ restricting to a triangle functor $\mathcal{C}_k^\circ \to \mathcal{C}_V^\circ$, and a triangle functor $\phi : \mathcal{C}_V^\circ \to \mathcal{C}_k$, as well as an element t in the centre of $\mathcal{C}_V^\circ$. We require that $\phi(t_{M\uparrow_k^V})$ is a split monomorphism with cone M for all objects M of $\mathcal{C}_k^\circ$.*
- *An object M of $\mathcal{C}_k^\circ$ degenerates to an object N of $\mathcal{C}_k^\circ$ if there is an object Q of $\mathcal{C}_V^\circ$ such that $\phi(cone(t_Q)) \simeq N$ and $p(Q) \simeq p(M \uparrow_k^V)$. We write $M \leq_{cdeg} N$ in this case.*

It is not hard to see that this generalises Yoshino's concept in case of stable categories for finite dimensional self-injective algebras. The main result is

Theorem 7. *Let $\mathcal{C}_k^\circ$ be a triangulated k-category with split idempotents. Then*

$$M \leq_{\Delta+nil} N \Rightarrow M \leq_{cdeg} N.$$

If $\mathcal{C}_k^\circ$ is the category of compact objects in an algebraic compactly generated triangulated k-category, then we get

$$M \leq_{\Delta+nil} N \Leftrightarrow M \leq_{cdeg} N.$$

4. Symmetry in the definition of the triangle degeneration

The definition of $\leq_{\text{Zwara}}$ and $\leq_\Delta$ bears some non-symmetry. Zwara proved in [9] for finite dimensional algebras A over a field and A-modules M and N that there is an A-module Z and a short exact sequence $0 \to Z \to Z \oplus M \to N \to 0$ if and only if there is an A-module Z' and a short exact sequence $0 \to N \to Z' \oplus M \to Z' \to 0$.

For the relation $\leq_\Delta$ we may pose the same question. Since the opposite category of a triangulated category is again triangulated, we denote by $\leq_{\Delta^{op}}$ the triangle relation between the corresponding objects, i.e. $M \leq_{\Delta^{op}} N$ if and only if there is Z' and a distinguished triangle $N \to M \oplus Z' \to Z' \to N[1]$.

Theorem 8. [5] *Let $\mathcal{T}$ be a triangulated k-category with split idempotents.*

- *If the endomorphism ring of each object in $\mathcal{T}$ is artinian, then $M \leq_{\Delta+nil} N \Leftrightarrow M \leq_{\Delta^{op}+nil} N$.*
- *If $\mathcal{T}_k$ is the category of compact objects in an algebraic compactly generated triangulated k-category, then $M \leq_{\Delta+nil} N \Leftrightarrow M \leq_{\Delta^{op}+nil} N$.*

We do not know if the statement is true for $\leq_\Delta$ instead of $\leq_{\Delta+\mathrm{nil}}$. The second statement uses our explicit construction of a degeneration and the construction of an explicit Q as in the definition of $\leq_{\mathrm{cdeg}}$ in the main theorem.

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An Alternate Approach to the Lie Bracket on Hochschild Cohomology

Sarah Witherspoon

(joint work with Lauren Grimley, Van C. Nguyen, Cris Negron)

Let k be a field, R a k-algebra, and $\otimes = \otimes_k$. Let $R^e = R \otimes R^{op}$ and let B denote the bar resolution of R as R^e-module. The Hochschild cohomology of R is $\mathrm{HH}^*(R) = \mathrm{H}^*(\mathrm{Hom}_{R^e}(B, R)) = \mathrm{Ext}^*_{R^e}(R, R)$.

Gerstenhaber [4] defined the graded Lie bracket on Hochschild cohomology at the chain level as follows. Let $f \in \mathrm{Hom}_{R^e}(B_i, R) \cong \mathrm{Hom}_k(R^{\otimes i}, R)$ and $g \in \mathrm{Hom}_{R^e}(B_j, R) \cong \mathrm{Hom}_k(R^{\otimes j}, R)$. Then $[f, g] = f \circ g - (-1)^{(i-1)(j-1)} g \circ f$ where

$$(f \circ g)(r_1 \otimes \cdots \otimes r_{i+j-1})$$

$$= \sum_{l=1}^{i} (-1)^{(j-1)(l-1)} f(r_1 \otimes \cdots \otimes r_{l-1} \otimes g(r_l \otimes \cdots \otimes r_{l+j-1}) \otimes r_{l+j} \otimes \cdots \otimes r_{i+j-1}).$$

With this bracket, and cup product, $\mathrm{HH}^*(R)$ is a Gerstenhaber algebra.

We will view the circle product $f \circ g$ above as a composition of functions:

$$B \xrightarrow{\Delta_B^{(2)}} B \otimes_R B \otimes_R B \xrightarrow{\mathbf{1}_B \otimes g \otimes \mathbf{1}_B} B \otimes_R B \xrightarrow{\phi_B} B \xrightarrow{f} R, \tag{1}$$

and $g \circ f$ will be defined similarly. The map $\Delta_B^{(2)}$ is given by $(\Delta_B \otimes \mathbf{1}_B)\Delta_B$ (which is equal to $(\mathbf{1}_B \otimes \Delta_B)\Delta_B$), where the diagonal map Δ_B is a chain map $\Delta_B : B \to B \otimes_R B$:

$$\Delta_B(r_0 \otimes \cdots \otimes r_{i+1}) = \sum_{j=0}^{i} (r_0 \otimes \cdots \otimes r_j \otimes 1) \otimes_R (1 \otimes r_{j+1} \otimes \cdots \otimes r_{i+1})$$

for all $r_0, \ldots, r_{i+1} \in R$. The definition of the map $\mathbf{1}_B \otimes g \otimes \mathbf{1}_B$ above includes signs so that on elements the map is given by $x \otimes y \otimes z \mapsto (-1)^{lj} x \otimes g(y) \otimes z$ for all $x \in B_l$, $y \in B_m$, $z \in B_n$. The map $\phi_B : B \otimes_R B \to B$ is a chain map of degree 1 given by

$$\begin{aligned} &\phi_B((1 \otimes r_1 \otimes \cdots \otimes r_{l-1} \otimes 1) \otimes_R r' \otimes_R (1 \otimes r_{l+j} \otimes \cdots \otimes r_{i+j-1} \otimes 1) \\ &\quad = (-1)^{l-1} \otimes r_1 \otimes \cdots \otimes r_{l-1} \otimes r' \otimes r_{l+j} \otimes \cdots \otimes r_{i+j-1} \otimes 1. \end{aligned}$$

Letting $\mu : B \to R$ be the natural quasi-isomorphism (that is, multiplication in degree 0 and the zero map in higher degrees), one checks that

$$d_B \phi_B + \phi_B d_{B \otimes_R B} = \mu \otimes 1 - 1 \otimes \mu.$$

In fact, any map ϕ_B satisfying the above equation also gives rise to Gerstenhaber's bracket on cohomology via the sequence of maps (1).

We next mimic this construction in the context of another resolution. We will require some conditions for our proof that the resulting bracket agrees with Gerstenhaber's bracket on cohomology: Let K be a projective resolution of R as an R^e-module for which there is an embedding into the bar resolution B that admits a section, and for which the diagonal map Δ_B restricts to a diagonal map Δ_K on K. Let $\mu_K : K \to R$ be the natural quasi-isomorphism. For example, the Koszul resolution of a Koszul algebra satisfies these conditions [3, 10].

Again, $\mu_K \otimes 1 - 1 \otimes \mu_K$ is a boundary in the complex $\mathrm{Hom}_{R^e}(K \otimes_R K, K)$. Let $\phi_K : K \otimes_R K \to K$ be a map for which $d_K \phi_K + \phi_K d_{K \otimes_R K} = \mu_K \otimes 1 - 1 \otimes \mu_K$. We now define a bracket at the chain level as before: Let $f \in \mathrm{Hom}_{R^e}(K_i, A)$, $g \in \mathrm{Hom}_{R^e}(K_j, A)$, and $\Delta_K^{(2)} = (\Delta_K \otimes 1)\Delta_K$ (this is equal to $\Delta_K^{(2)} = (1 \otimes \Delta_K)\Delta_K$ since Δ_K is induced by Δ_B). Define $f \circ_{\phi_K} g$ to be the composition (1) in which B, $\Delta_B^{(2)}$, and ϕ_B are replaced by K, $\Delta_K^{(2)}$, and ϕ_K, respectively. Define $[f, g]_{\phi_K}$ to be $f \circ_{\phi_K} g - (-1)^{(i-1)(j-1)} g \circ_{\phi_K} f$.

Theorem 1. [8] *The operation $[\cdot, \cdot]_{\phi_K}$ induces the Lie bracket on $\mathrm{HH}^*(R)$.*

One advantage to this approach in defining brackets is that the chain maps between K and B are only needed to check the required conditions. They need not be used in any explicit computations once these conditions are known to hold. One still may need to find and use the map ϕ_K; in practice, this can be easier and more illuminating than finding and using explicit chain maps between B and K. A

formula for ϕ_K in terms of a contracting homotopy for K is given in [8, Section 3.3]. A disadvantage to this approach is that not all resolutions K satisfy the required conditions. There are slightly weaker conditions given in [8, Section 3.4].

We give some examples and applications of this alternate approach to Gerstenhaber's bracket: In [8] with Negron, we recover the known graded Lie structure on $\mathrm{HH}^*(R)$ for $R = S(V)$, a symmetric algebra on a finite dimensional vector space V, and for $R = kG$, G a cyclic group of prime order p where $\mathrm{char}(k) = p$. For $S(V)$, we see the expected identification of the bracket with the Schouten bracket on polyvector fields on affine space. For kG, the bracket was computed more generally for cyclic groups by Sanchez-Flores [11]. In [9] with Negron, we gave some general structure results for the Hochschild cohomology of $S(V) \rtimes G$, where G is a finite group with representation V; these results were presented in Negron's talk, and we refer to his abstract for details.

With Grimley and Nguyen in [6], we computed new examples of Gerstenhaber brackets: Let $\Lambda_q = k\langle x, y\rangle/(x^2, y^2, xy + qyx)$, where q is a nonzero scalar. Buchweitz, Green, Madsen, and Solberg [2] computed the algebra structure of $\mathrm{HH}^*(\Lambda_q)$ under cup product; in case q is not a root of unity, Λ_q has infinite global dimension yet has finite dimensional Hochschild cohomology (which answered a question of Happel).

In [6] we exploited the fact that Λ_q is a twisted tensor product (definition below) of its subalgebras generated by x and by y. A resolution K may be defined as a twisted tensor product of resolutions for these subalgebras, and the map ϕ_K (in (1)) may be expressed in terms of such maps for the factors. We used this map to compute explicitly brackets for all nonzero scalars q, describing the Lie algebra structure of $\mathrm{HH}^1(\Lambda_q)$ in the different cases (often it is abelian) and the structure of $\mathrm{HH}^*(\Lambda_q)$ as a module over $\mathrm{HH}^1(\Lambda_q)$. In [6] we also proved a general structure result for the Hochschild cohomology of a twisted tensor product, as we state next, after defining twisted tensor product.

Let R, S be k-algebras graded by abelian groups A, B. Let $t : A \otimes_{\mathbb{Z}} B \to k^\times$ be a homomorphism of abelian groups, denoted $t(a \otimes_{\mathbb{Z}} b) = t^{\langle a|b\rangle}$ for all $a \in A$, $b \in B$. The twisted tensor product of R and S is $R \otimes^t S = R \otimes S$ as a vector space, and $(r \otimes s) \cdot^t (r' \otimes s') = t^{\langle |r'| | |s| \rangle} rr' \otimes ss'$ for all homogeneous $r, r' \in R$ and $s, s' \in S$, where $|r'|$, $|s|$ are the degrees of r', s in A, B (see [1]).

Assume at least one of R, S is finite dimensional. The following theorem extends [1, Theorem 4.7] of Bergh and Oppermann who first gave the isomorphism below as an isomorphism of associative algebras. It was proved to be an isomorphism of Gerstenhaber algebras in the untwisted tensor product case by Le and Zhou [7]. In our proof, we use the alternate approach to brackets as described above.

Theorem 2. [6] *There is an isomorphism of Gerstenhaber algebras*

$$\mathrm{HH}^{*,A'}(R) \otimes \mathrm{HH}^{*,B'}(S) \cong \mathrm{HH}^{*,A' \oplus B'}(R \otimes^t S),$$

where the Gerstenhaber bracket on the left side is given by [7, Prop.-Defn. 2.2]*, and* $A' = \bigcap_{b \in B} \mathrm{Ker}\, t^{\langle \cdot | b\rangle}$, $B' = \bigcap_{a \in A} \mathrm{Ker}\, t^{\langle a | \cdot \rangle}$.

Many of the algebras Λ_q discussed above provide nontrivial illustrations of Theorem 2.

As a final application, we mention Grimley's PhD thesis [5], containing general formulas for the Gerstenhaber algebra structure of the Hochschild cohomology of

$$\Lambda_{\mathbf{q}}^{(2,\dots,2)} = k\langle x_1,\dots,x_n\rangle/(x_i^2,\ x_ix_j + q_{ij}x_jx_i) \quad \text{and} \quad \Lambda_{\mathbf{q}}^{(2,\dots,2)} \rtimes G,$$

where $\mathbf{q}$ is a set of nonzero scalars q_{ij} $(i < j)$, and G is a finite group acting diagonally on the set $x_1,\dots,x_n$, inducing an algebra automorphism on $\Lambda_q^{(2,\dots,2)}$.

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The Gerstenhaber bracket as a Schouten bracket for polynomial rings extended by finite groups

Cris Negron

(joint work with Sarah Witherspoon)

We discuss here the work of [7], in which we describe the Gerstenhaber bracket on the Hochschild cohomology of a smash product $S(V)\#G$ between a polynomial ring $S(V)$ (written here as the symmetric algebra of a vector space V) and a finite group G acting linearly on V. We work over a field k of characteristic 0, and let $HH^\bullet(A)$ denote the Hochschild cohomology of an algebra A

One can understand the significance of the smash product via its category of modules. Algebraically, modules over the $S(V)\#G$ are modules over $S(V)$ with a compatible G-action, in the most obvious manner, while $S(V)\#G$-modules can be understood geometrically as G-equivariant sheaves on affine space, or rather

quasi-coherent sheaves on the orbifold $[\mathbb{A}^n/G]$.[1] Whence we may identify, via the work of Lowen and Van den Bergh [5], the Hochschild cohomology of the smash product with the Hochschild cohomology of the quotient $[\mathbb{A}^n/G]$.

Motivated by the classic Hochschild-Kostant-Rosenberg theorem [4], which we review below, we seek a geometric description of the Hochschild cohomology of the smash product $S(V)\#G$, along with the Gerstenhaber bracket. Such a description is achieved in [7] and relayed here in Section 1.2.

We note that the work [7] comes after initial contributions of Shepler-Witherspoon and Halbout-Tang [9, 3], and extends results therein, and that we rely on techniques developed in the related work [6].

1.1. **Review of the HKR theorem.** Consider a smooth k-scheme $\mathscr{M}$ and let $k[\mathscr{M}]$ denote the algebra of global functions on $\mathscr{M}$. Let

$$T^{poly}_{\mathscr{M}} = \bigwedge^{\bullet}_{k[\mathscr{M}]} T_{\mathscr{M}}$$

denote the associated algebra of polyvector fields, where $T_{\mathscr{M}}$ denotes the global section of the tangent sheaf on $\mathscr{M}$, i.e. global vector fields on $\mathscr{M}$. We consider $T^{poly}_{\mathscr{M}}$ to be graded with $T_{\mathscr{M}}$ concentrated in degree 1.

The standard bracket on vector fields $T_{\mathscr{M}}$ extends uniquely to a graded Lie structure on the shifted space $\Sigma T^{poly}_{\mathscr{M}}$ so that $T^{poly}_{\mathscr{M}}$ becomes a, so called, *Gerstenhaber algebra*. This canonical graded Lie bracket on $T^{poly}_{\mathscr{M}}$ is called the *Schouten bracket*.

The following result is due essentially to Hochschild, Kostant, and Rosenberg [4].

The HKR theorem. *Let $\mathscr{M}$ be a smooth k-scheme. Then there is a canonical identification of Gerstenhaber algebras $HH^{\bullet}(k[\mathscr{M}]) = T^{poly}_{\mathscr{M}}$, where the cohomology $HH^{\bullet}(k[\mathscr{M}])$ is given its Gerstenhaber bracket and $T^{poly}_{\mathscr{M}}$ is given the Schouten bracket.*

We seek, in general, an analog of the HKR theorem for orbifold quotients $[\mathscr{M}/G]$, and begin here with a study of quotients of affine space by finite linear group actions.

1.2. **The cohomology $HH^{\bullet}(S(V)\#G)$ and polyvector fields.** Given a smash product $S(V)\#G$, as above, we have canonical embeddings and projections

$$HH^{\bullet}(S(V)\#G) \to \left(\bigoplus\nolimits_{g\in G} T^{poly}_{\mathbb{A}^n} g\right)^G \xrightarrow{p} HH^{\bullet}(S(V)\#G), \tag{1}$$

where the "g"s in the expression $T^{poly}_{\mathbb{A}^n} g$ are simply labels, and the implicit G-action on the space $\bigoplus_{g\in G} T^{poly}_{\mathbb{A}^n} g$ is induced by the standard G-action on polyvector fields and the adjoint action of G on itself. We have a naïve extension of the Schouten

[1] By "orbifold" here we mean simply the stack quotient. The term orbifold refers to the fact that we are taking the quotient by a finite group.

bracket to the space $\bigoplus_{g\in G} T^{poly}_{\mathbb{A}^n} g$ and invariants by the formula

$$[\sum_{g\in G} X_g g, \sum_{h\in G} Y_h h]_{\text{Sch.ext}} := \sum_{g,h\in G} [X_g, Y_h]_{\text{Sch}}\ gh,$$

where $X_g, Y_h \in T^{poly}_{\mathbb{A}^n}$ and the non-subscripted g, h are labels as in (1).

The situation (1) follows from a vector space description of the Hochschild cohomology given in the works of Ginzburg-Kaledin and Farinati [2, 1],

$$HH^\bullet(S(V)\#G) = \left(\bigoplus_{g\in G} T^{poly}_{(\mathbb{A}^n)^g} \det\left(T^{\perp}_{(\mathbb{A}^n)^g}\right)\right)^G.$$

We state our main result in terms of the relationship (1).

Theorem 1 ([7])**.** *For classes $\sum_g X_g g, \sum_h Y_h h$ in the cohomology $HH^\bullet(S(V)\#G)$, which we view as a subspace in $\bigoplus_{g\in G} T^{poly}_{\mathbb{A}^n} g$, the Gerstenhaber bracket is given by the formula*

$$[\sum_{g\in G} X_g g, \sum_{h\in G} Y_h h]_{\text{Gerst}} := \sum_{g,h\in G} p[X_g, Y_h]_{\text{Sch}}\ gh,$$

where p is the projection from (1)*.*

We note that the projection p is completely canonical, and can be defined in a much more general geometric context. The above theorem was achieved for abelian G in the work of Halbout and Tang [3].

In order to give some corollaries we introduce a few more notations. Note that the Hochschild cohomology has a canonical decomposition

$$HH^\bullet(S(V)\#G) = \bigoplus_{i=1}^{\dim(V)} \mathscr{D}(i), \tag{2}$$

where $\mathscr{D}(i)$ is the collection of polyvector fields labeled by group elements for which the fixed space V^g is of codimension i. The first possibly nonzero class in each $\mathscr{D}(i)$ will occur in degree i.

Corollary 2 ([7])**.** *(1) Let $X \in \mathscr{D}(i)$ and $Y \in \mathscr{D}(j)$ be classes of respective degrees i and j. Then $[X, Y]_{\text{Gerst}} = 0$.*

(2) The Hochschild cohomology is a graded Gerstenhaber algebra under the codimension grading (2)*, i.e. the cup product and bracket satisfy $\mathscr{D}(i) \cdot \mathscr{D}(j) \subset \mathscr{D}(i+j)$ and $[\mathscr{D}(i), \mathscr{D}(j)] \subset \mathscr{D}(i+j)$.*

The result (1) generalizes a vanishing result of [9], which applied originally to classes in degree 2, and (2) was pointed out to us by Travis Schedler. The fact that the Hochschild cohomology is graded, as an algebra, by the codimension grading was already known [8], while the fact that the bracket respects the codimension grading is new.

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Koszul Calculus

Andrea Solotar
(joint work with Roland Berger, Thierry Lambre)

The idea of this joint work with Ronad Berger and Thierry Lambre is to present a calculus which is well-adapted to quadratic algebras. This calculus is defined in Koszul cohomology (homology) by cup products (cap products). Koszul homology and cohomology are interpreted in terms of derived categories. If the algebra is not Koszul, Koszul (co)homology provides different information than Hochschild (co)homology. Koszul homology is related to de Rham cohomology. If the algebra is Koszul, Koszul cohomology is related to the Calabi-Yau property. The calculus is made explicit on a non-Koszul example.

Quadratic algebras are associative algebras defined by homogeneous quadratic relations. Since their definition by Priddy [7], Koszul algebras form a widely studied class of quadratic algebras [6]. In his monograph [5], Manin brings out a general approach of quadratic algebras (not necessarily Koszul), including the fundamental observation that quadratic algebras form a category which should be a relevant framework for a noncommutative analogue of projective algebraic geometry. According to this general approach, non-Koszul quadratic algebras deserve certainly more attention.

The goal here is to introduce new general tools for studying quadratic algebras. These tools consist in a (co)homology, called Koszul (co)homology, together with products, called Koszul cup and cap products. They are organized in a calculus, called Koszul calculus. If two quadratic algebras are isomorphic in the sense of the Manin category, their Koszul calculus are isomorphic. If the quadratic algebra

is Koszul, the Koszul calculus is isomorphic to Hochschild (co)homology endowed with usual cup and cap products – called Hochschild calculus. .

The Koszul homology $HK_\bullet(A, M)$ of a quadratic algebra A with coefficients in a bimodule M is defined by applying the functor $M \otimes_{A^e} -$ to the Koszul complex of A, analogously for the Koszul cohomology $HK^\bullet(A, M)$. If A is Koszul, the Koszul complex is a projective resolution of A, so that $HK_\bullet(A, M)$ (resp. $HK^\bullet(A, M)$) is isomorphic to the Hochschild homology $HH_\bullet(A, M)$ (resp. Hochschild cohomology $HH^\bullet(A, M)$). Restricting the Koszul calculus to $M = A$, we present a non-Koszul quadratic algebra A which is such that $HK_\bullet(A) \ncong HH_\bullet(A)$ and $HK^\bullet(A) \ncong HH^\bullet(A)$. So $HK_\bullet(A)$ and $HK^\bullet(A)$ provide further invariants associated to the Manin category, besides those provided by Hochschild (co)homology. We have proven that Koszul homology (cohomology) is isomorphic to a Hochschild hyperhomology (hypercohomology), showing that this new homology (cohomology) becomes natural in terms of derived categories.

For any unital associative algebra A, the Hochschild cohomology of A with coefficients in A itself, endowed with the cup product, has a richer structure provided by Gerstenhaber product $\circ$, called Gerstenhaber calculus [1]. When $\circ$ is replaced in the structure by the graded bracket associated to $\circ$, that is, the Gerstenhaber bracket $[-,-]$, the calculus becomes a Gerstenhaber algebra [1]. Next, the Gerstenhaber algebra and the Hochschild homology of A, endowed with cap products, are organized in a Tamarkin-Tsygan calculus [8], see also [4]. In the Tamarkin-Tsygan calculus, the Hochschild differential b is defined from the multiplication μ of A and the Gerstenhaber bracket by

$$b(f) = [\mu, f] \tag{1}$$

for any Hochschild cochain f.

The obstruction to see the Koszul calculus as a Tamarkin-Tsygan calculus is the following: the Gerstenhaber product $\circ$ *does not make sense on Koszul cochains.* However, this negative answer can be bypassed by the fundamental formula of the Koszul calculus

$$b_K(f) = -[e_A, f]_{\underset{K}{\smile}} \tag{2}$$

where b_K is the Koszul differential, e_A is the fundamental 1-cocycle and f is any Koszul cochain.

In formula (2), $[-,-]_{\underset{K}{\smile}}$ is the graded bracket associated to the Koszul cup product $\underset{K}{\smile}$. In other words, *the Koszul differential may be defined from the Koszul cup product.* Therefore, the Koszul calculus is simpler than the Tamarkin-Tsygan calculus, since no additional product such as $\circ$ is required to express the differential by means of a graded bracket. The Koszul calculus is more flexible since the formula (2) is valid for any bimodule M, while the definitions of Gerstenhaber product and bracket are meaningless when considering other bimodules of coefficients [2]; it is also more symmetric since there is an analogue of (2) in homology, where the Koszul cup product is replaced by the Koszul cap product.

In [3], Ginzburg mentions that the Hochschild cohomology algebras of A and its Koszul dual $A^!$ are isomorphic if the quadratic algebra A is Koszul. As an application of Koszul calculus, we obtain such a Koszul duality theorem linking the Koszul cohomology algebras of A and $A^!$ *for any quadratic algebra A, Koszul or not*. So the true nature of the Koszul duality theorem is independent of any assumption on quadratic algebras. The proof of our result lies on a Koszul duality at the level of Koszul cochains and uses standard facts on duality of finite dimensional vector spaces.

In the Tamarkin-Tsygan calculus, the Connes differential B defined on Hochschild homology is an essential ingredient. Although B does not send Koszul chains to Koszul chains, we have solved the question to find such a differential at the level of Koszul homology classes for some very particular cases, the general case being open. An important role is played by the Rinehart-Goodwillie operator whose Koszul analogue is the left Koszul cap product by the fundamental 1-cocycle e_A, leading to the higher Koszul homology of A. We conjecture that a quadratic algebra A is Koszul if and only if its higher Koszul homology annihilates in positive degrees.

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Hochschild cohomology of a smash product with a cyclic group

Yuri Volkov

1. The general problem

Let us fix some field $\mathbf{k}$ and an associative unital $\mathbf{k}$-algebra A. We write simply $\otimes$ instead of $\otimes_{\mathbf{k}}$. Also we fix some finite group G with homomorphism $\eta : G \to \operatorname{Aut} A$. We write simply ${}^{\alpha}a$ instead of $\eta(\alpha)(a)$ for $\alpha \in G$ and $a \in A$.

We define the smash product $A\#\mathbf{k}G$ in the following way. The algebra $A\#\mathbf{k}G$ is isomorphic to $A \otimes \mathbf{k}G$ as a vector space. The multiplication of the elements $a \otimes \alpha, b \otimes \beta \in A\#\mathbf{k}G$ is defined by the formula

$$(a \otimes \alpha)(b \otimes \beta) = a^{\alpha} b \otimes \alpha\beta.$$

For simplicity we write $a\alpha$ instead of $a \otimes \alpha$ and AG instead of $A\#\mathbf{k}G$. We fix also some AG-bimodule M. Note that G acts on M by the rule ${}^{\alpha}x = \alpha x \alpha^{-1}$ for $\alpha \in G$ and $x \in M$. This action and the action of G on A induce an action of G on $\mathrm{HH}^*(A, M)$.

It is well known (see, for example, [1], [2] and [3]) that in this case there is a spectral sequence

$$E_2^{i,j} = \mathrm{H}^i(G, \mathrm{HH}^j(A, M)) \Rightarrow \mathrm{HH}^{i+j}(AG, M) \tag{1}$$

From now we assume that there is some map $\mu_M : M \otimes M \to M$, which defines a structure of unital associative algebra on M and induces an AG-bimodule homomorphism from $M \otimes_{AG} M$ to M. Such μ_M induces a graded algebra structure on $\mathrm{HH}^*(AG, M)$. Then the spectral sequence (1) is a spectral sequence of algebras. The multipliation on the seond page is simply cup product on the cohomology of G with coefficients in the $\mathbf{k}G$-module algebra $\mathrm{HH}^*(A, M)$.

If $\mathbf{k}$ is a field such that $\mathrm{char}\,\mathbf{k} \nmid |G|$, then the spectral sequence (1) collapses at the second page and we obtain isomorphism of algebras $\mathrm{HH}^*(A, M)^G \cong \mathrm{HH}^*(AG, M)$. This result is good enough, but the situation becomes more difficult if $\mathrm{char}\,\mathbf{k} \mid |G|$. So the question considered here is

Question 1. *How the algebras* $\mathrm{HH}^*(A, M)^G$ *and* $\mathrm{HH}^*(AG, M)$ *are related in the case where* $\mathrm{char}\,\mathbf{k} \mid |G|$*?*

2. Results

The results presented here and in the next section are obtained by the author of the present text in the joint work with E. Marcos. Assume that $\mathrm{char}\,\mathbf{k} = p > 0$ and G is a cyclic group of order q, where q is some power of p (in fact our methods work for such group G that has a normal subgroup H such that $p \nmid |H|$ and $G/H = C_q$). We say that a spectral sequence E is (R, S)-degenerated if $E_R^{i,j} = 0$ for $i \geq S$. Then the following theorem is true.

Theorem 1. $\dim_{\mathbf{k}} \mathrm{HH}^n(AG, M) \geq \dim_{\mathbf{k}} \mathrm{HH}^n(A, M)^G$ *for any* $n \geq 0$*. Moreover, the following conditions are equivalent:*

1. *the sequence* (1) *is* $(3, 2)$*-degenerated;*
2. $\dim_{\mathbf{k}} \mathrm{HH}^n(AG, M) = \dim_{\mathbf{k}} \mathrm{HH}^n(A, M)^G$ *for any* $n \geq 0$*;*
3. $\dim_{\mathbf{k}} \mathrm{HH}^1(AG, M) = \dim_{\mathbf{k}} \mathrm{HH}^1(A, M)^G$.

From here on we consider the case of $(3, 2)$-degeneration. Let ρ denote some generator of C_q. If $\mathcal{A}$ is a graded algebra and $D : \mathcal{A} \to \mathcal{A}$ is a graded derivation, then we define the graded algebra $\mathcal{A}[x, D]$ in the following way. Its underlining graded vector space is the space $\mathcal{A}[x]$ with n-th component generated by the elements of the form ax^i where $a \in \mathcal{A}_{n-i}$. The multiplication of two elements from $\mathcal{A} \subset \mathcal{A}[x, D]$ or two elements from $\mathbf{k}[x] \subset \mathcal{A}[x, D]$ is defined as usual. We define the left multiplication of $a \in \mathcal{A}_i$ by the elemet x by the equality $xa = (-1)^i ax + D(a)$. Let W be the ideal of $\mathrm{HH}^*(A, M)^G$ defined by the equality $W = (1 - \rho)^{q-1} \mathrm{HH}^*(A, M)$.

A more or less good answer to Question 1 in the case of $(3,2)$-degenerating is given by the following theorem.

Theorem 2. *There is a graded subalgebra $\mathcal{A}$ of $\mathrm{HH}^*(A,M)^G$ containing W such that $\mathrm{HH}^*(A,M)^G/W = (\mathcal{A}/W)[x,D]/\langle x^2 - a\rangle$ for some graded derivation D of $\mathcal{A}$ and some $a \in \mathcal{A}$. At the same time there is a graded ideal X of $\mathrm{HH}^*(AG,M)$ such that $X^2 = 0$ and $\mathrm{HH}^*(AG,M)/X \cong \mathcal{A}$. Moreother, there are two filtrations of graded $\mathcal{A}$-bimodules*

$$0 = X_{q-1} \subset X_{q-2} \subset \cdots \subset X_1 \subset X_0 = X$$

and

$$0 = Y_{q-1} \subset Y_{q-2} \subset \cdots \subset Y_1 \subset Y_0 = (\mathcal{A}/W)[-1]$$

such that $Y_i/Y_{i+1} \cong X_{q-2-i}/X_{q-1-i}$ for $0 \le i \le q-2$. Moreover, if $(1-\rho)^i \mathrm{HH}^*(A,M)^H = \mathrm{HH}^*(A,M)^G$ for some $0 \le i \le q-1$, then $X_{q-1-i} = 0 = Y_{i+1}$ and $X = X_{q-2-i} \cong Y_i = (\mathcal{A}/W)[-1]$.

Remark 1. *If $\mathrm{HH}^*(A,M)^G$ is graded commutative, then $D = 0$. If additionally $p \neq 2$, then we have also $a = 0$.*

3. Application

Let $\mathcal{R}$ be a finite dimensional $\mathbf{k}$-algebra. We denote by $D\mathcal{R}$ the $\mathcal{R}$-bimodule $\mathrm{Hom}_{\mathbf{k}}(\mathcal{R},\mathbf{k})$. Let $T\mathcal{R}$ be a trivial singular extension of $\mathcal{R}$ by $D\mathcal{R}$, i.e. $T\mathcal{R}$ is an algebra, whose underlining space is $\mathcal{R} \oplus D\mathcal{R}$ and the multiplication is defined by the equality $(a,\hat{u})(b,\hat{v}) = (ab, a\hat{v} + \hat{u}b)$ for $a,b \in \mathcal{R}$ and $\hat{u},\hat{v} \in D\mathcal{R}$. There is a C_n-grading on $T\mathcal{R}$ such that $(T\mathcal{R})_0 = \mathcal{R}$ and $(T\mathcal{R})_1 = D\mathcal{R}$. This grading induces a C_n-grading on $\mathrm{HH}^*(T\mathcal{R})$.

Let $\mathcal{R}_n$ denotes the algebra, whose underlining space is $(\mathcal{R} \oplus D\mathcal{R})^n$ and the multiplication is defined by the equality

$$(a_1,\hat{u}_1,\ldots,a_n,\hat{u}_n)(b_1,\hat{v}_1,\ldots,b_n,\hat{v}_n) = (a_1b_1, a_1\hat{v}_1 + \hat{u}_1b_2, \ldots, a_nb_n, a_n\hat{v}_n + \hat{u}_nb_1)$$

for $a_1,\ldots,a_n,b,\ldots,b_n \in \mathcal{R}$ and $\hat{u}_1,\ldots,\hat{u}_n,\hat{v}_1,\ldots,\hat{v}_n \in D\mathcal{R}$. There is an action of $C_n = \langle \rho \mid \rho^n = 1\rangle$ on $\mathcal{R}_n$ defined by the equality

$$\rho(a_1,\hat{u}_1,\ldots,a_n,\hat{u}_n) = (a_2,\hat{u}_2,\ldots,a_n,\hat{u}_n,a_1,\hat{u}_1).$$

Since C_n acts on $\mathcal{R}_n$ by powers of Nakayama automorphism, it follows from [4, Corollary 2] that $\mathrm{HH}^*(\mathcal{R}_n) = \mathrm{HH}^*(\mathcal{R}_n)^{C_n}$. In particular, if $\mathrm{char}\,\mathbf{k} \nmid n$, then $\mathrm{HH}^*(\mathcal{R}_n) \cong \mathrm{HH}^*(T\mathcal{R})_0$. We can apply our results to obtain the following theorem.

Theorem 3. *Let $\mathcal{R}$ be a finite dimensional $\mathbf{k}$-algebra and $n \ge 1$ be some integer such that $\mathrm{char}\,\mathbf{k} \mid n$. Then there is a graded algebra $\mathcal{A}_n$ such that $\mathrm{HH}^*(\mathcal{R}_n)$ is a trivial singular extension of $\mathcal{A}_n$ by $\mathcal{A}_n[-1]$ and $\mathrm{HH}^*(T\mathcal{R})_0$ is some singular extension of $\mathcal{A}_n$ by $\mathcal{A}_n[-1]$. In other words,*

$$\mathrm{HH}^*(\mathcal{R}_n) \cong \mathcal{A}_n[x,0]/x^2$$

and there is an exact sequence of $\mathcal{A}_n$-modules

$$\mathcal{A}_n[-1] \rightarrowtail \mathrm{HH}^*(T\mathcal{R})_0 \twoheadrightarrow \mathcal{A}_n,$$

which becomes an exact sequence of algebras if we equip $\mathcal{A}_n[-1]$ with zero multiplication.

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Singular Hochschild cohomology and Gerstenhaber algebra structure

Zhengfang Wang

Let k be a commutative ring and A be a k-algebra such that A is projective as a k-module. Recall that for any $m \in \mathbb{Z}$, the Hochschild cohomology group $\mathrm{HH}^m(A, A)$ can be viewed as the Hom-space from A to $A[m]$ in the bounded derived category $\mathcal{D}^b(A \otimes_k A^{\mathrm{op}})$ of the enveloping algebra $A \otimes A^{\mathrm{op}}$. Namely, we have

$$\mathrm{HH}^m(A, A) = \mathrm{Hom}_{\mathcal{D}^b(A \otimes_k A^{\mathrm{op}})}(A, A[m]).$$

In particular, we have that the negative part $\mathrm{HH}^{<0}(A, A)$ vanishes. From this motivation, we replace the bounded derived category $\mathcal{D}^b(A \otimes_k A^{\mathrm{op}})$ by the singular category $\mathcal{D}_{\mathrm{sg}}(A \otimes A^{\mathrm{op}})$ [1, 2], which is the Verdier quotient of the bounded derived category $\mathcal{D}^b(A \otimes A^{\mathrm{op}})$ by the full subcategory $\mathcal{K}^b(A \otimes A^{\mathrm{op}}\text{-proj})$ consisting of perfect complexes of A-modules and define the *singular Hochschild cohomology* (denoted by $\mathrm{HH}^m_{\mathrm{sg}}(A, A)$) of degree m as the Hom-space from A to $A[m]$ in the singular category $\mathcal{D}_{\mathrm{sg}}(A \otimes A^{\mathrm{op}})$. Namely, we define

$$\mathrm{HH}^m_{\mathrm{sg}}(A, A) := \mathrm{Hom}_{\mathcal{D}_{\mathrm{sg}}(A \otimes A^{\mathrm{op}})}(A, A[m]).$$

Different from the Hochschild cohomology $\mathrm{HH}^*(A, A)$, we note that in general $\mathrm{HH}^{<0}_{\mathrm{sg}}(A, A)$ does not vanish.

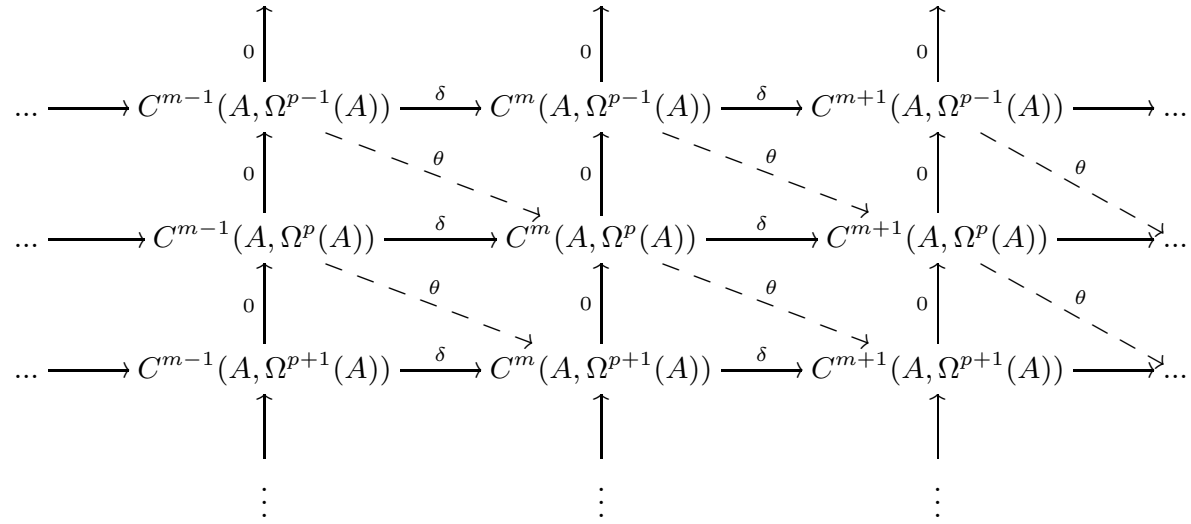
Denote by $\Omega^p(A)$ the kernel of the differential $d_{p-1} : A^{\otimes p+1} \to A^{\otimes p}$ in the bar resolution $\mathrm{Bar}_*(A)$. Denote $C^m(A, \Omega^p(A)) := \mathrm{Hom}_k(A^{\otimes m}, \Omega^p(A)$. Construct a k-linear map

$$\theta_{m,p} : C^m(A, \Omega^p(A)) \to C^{m+1}(A, \Omega^{p+1}(A))$$

as follows, for any $f \in C^m(A, \Omega^p(A))$,

$$\theta_{m,p}(f)(a_1 \otimes a_2 \otimes \cdots \otimes a_{m+1}) := (-1)^p d(f(a_1 \otimes \cdots \otimes a_m) \otimes a_{m+1} \otimes 1)$$

where $a_1 \otimes \cdots \otimes a_{m+1} \in A^{\otimes m+1}$ and d represents the differential in the bar resolution $\mathrm{Bar}_*(A)$. Then we have the following double complex $C^*(A, \Omega^*(A))$:



We denote the colimit of the inductive system

$$C^{m-1}(A, \Omega^{p-1}(A)) \xrightarrow{\theta_{m-1,p-1}} C^m(A, \Omega^p(A)) \xrightarrow{\theta_{m,p}} \cdots$$

by $C^{m-p}_{\mathrm{sg}}(A, A)$. That is, for any $m \in \mathbb{Z}$,

$$C^m_{\mathrm{sg}}(A, A) := \varinjlim_{\substack{r \in \mathbb{Z}_{\geq 0} \\ m+r \geq 0}} C^{m+r}(A, \Omega^r(A)).$$

So we obtain a complex $C^*_{\mathrm{sg}}(A, A)$ (called *singular Hochschid cochain complex*)

$$\cdots \longrightarrow C^{m-1}_{\mathrm{sg}}(A, A) \xrightarrow{\overline{\delta}^{m-1}} C^m_{\mathrm{sg}}(A, A) \xrightarrow{\overline{\delta}^m} C^{m+1}_{\mathrm{sg}}(A, A) \longrightarrow \cdots,$$

where the differential $\overline{\delta}$ is induced from the Hochschild differential in the double complex above. Then we have the following result.

Theorem 1. *(1) For any $m \in \mathbb{Z}$, $H^m(C^*_{\mathrm{sg}}(A, A)) \cong \mathrm{HH}^m_{\mathrm{sg}}(A, A)$.*

(2) There is a generalized Gerstenhaber bracket $[\cdot, \cdot]$ defined in the total complex $\mathrm{Tot}(C^*(A, \Omega^*(A)))$ which makes $C^*_{\mathrm{sg}}(A, A)$ into a differential graded Lie algebra.

(3) $\mathrm{HH}^*_{\mathrm{sg}}(A, A)$ is a Gerstenhaber algebra with the Yonede-product and the generalized Gerstenhaber bracket $[\cdot, \cdot]$ above.

We give a PROP interpretation for the generalized Gerstenhaber bracket in the total complex $\mathrm{Tot}(C^*(A, \Omega^*(A)))$. For any $m, p \in \mathbb{Z}_{\geq 0}$, we denote $\mathcal{P}_A(m, p) := C^m(A, \Omega^{p-1}(A))$, here we use the notation $\Omega^{-1}(A) := k$. Then we have the following results.

Theorem 2. *Let $\mathcal{P}$ be a (non-symmetric) PROP over k. Then there is a $\mathbb{Z}$-graded Lie algebra structure on the total space*

$$\mathcal{P}_{>0} := \bigoplus_{m,p\in\mathbb{Z}_{>0}} \mathcal{P}(m,p)$$

where the $\mathbb{Z}$-grading is defined as follows: for $n \in \mathbb{Z}$,

$$(\mathcal{P}_{>0})_n := \bigoplus_{\substack{m,p\in\mathbb{Z}_{>0}\\ m-p=n}} \mathcal{P}(m,p).$$

Theorem 3 (Joint-work with Guodong Zhou)**.** *$\mathcal{P}_A$ is indeed a PROP and the two Lie brackets on $\mathcal{P}_A$ coincide.*

At last, we have the invariance of the Gerstenhaber algebra structure in $\mathrm{HH}^*_{\mathrm{sg}}(A,A)$ under singular equivalences of Morita type with level [3] or derived equivalences.

Theorem 4. *Let k be a field. Let A and B be two finite-dimensional k-algebras. Suppose that a paris of bimodules $({}_AM_B, {}_BN_A)$ defines a singular equivalence of Morita type with level $l \in \mathbb{Z}_{\geq 0}$ between A and B. Then the functor $[l] \circ (M \otimes_B - \otimes_B N)$ induces an isomorphism of Gerstenhaber algebras*

$$[l] \circ (M \otimes_B - \otimes_B N) : \mathrm{HH}^*_{\mathrm{sg}}(B,B) \to \mathrm{HH}^*_{\mathrm{sg}}(A,A).$$

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Batalin-Vilkovisky structures in Hochschild cohomology and Poisson cohomology

Guodong Zhou

This report is an extended version of the abstract of the talk I gave in this Oberwolfach meeting. It summarizes a small amount of known results, by no means all, about Batalin-Vilkovisky structure in Hochschild cohomology and Poisson cohomology.

1. Gerstenhaber algebras, differential calculi and Batalin-Vilkovisky algebras

A *Gerstenhaber algebra* over a field k is a $\mathbb{N}$-graded vector space together with a cup product and a Lie bracket of degree 1 such that it is a graded commutative algebra via the cup product and that it becomes a graded Lie algebra via the Lie bracket, and furthermore, they satisfies a compatibility condition.

A *differential calculus* is the data $(\mathcal{H}^*, \cup, [\ ,\ ], \mathcal{H}_*, \cap, B)$ of $\mathbb{N}$-graded vector spaces satisfying the following properties:

(i) $(\mathcal{H}^*, \cup, [\ ,\ ])$ is a Gerstenhaber algebra;

(ii) $\mathcal{H}_*$ is a graded module over $(\mathcal{H}^*, \cup)$ via the map $\cap : \mathcal{H}_r \otimes \mathcal{H}^p \to \mathcal{H}_{r-p}, z \otimes \alpha \mapsto z \cap \alpha$ for $z \in \mathcal{H}_r$ and $\alpha \in \mathcal{H}^p$. That is, if we denote $\iota_\alpha(z) = (-1)^{rp} z \cap \alpha$, then $\iota_{\alpha \cup \beta} = \iota_\alpha \iota_\beta$;

(iii) There is a map $B : \mathcal{H}_* \to \mathcal{H}_{*+1}$ such that $B^2 = 0$ and we have the Cartan relation

$$[L_\alpha,\ \iota_\beta]_{gr} = (-1)^{|\alpha|-1} \iota_{[\alpha,\beta]}$$

where we denote

$$L_\alpha = [B, \iota_\alpha]_{gr} = B\iota_\alpha - (-1)^{|\alpha|} \iota_\alpha B.$$

Many (co)homological theories give rise to differential calculi.

Let A be an associative algebra over a field k. The Hochschild cohomology $HH^*(A)$ of A is a Gerstenhaber algebra [5] via the cup product and the Gerstenhaber Lie bracket. Hochschild cohomogy acts on Hochschild homology via the cap product and the Lie derivative, which together with the usual Connes' differential over Hochschild homology form a differential calculus; see [4][12].

Let S be a Poisson algebra. Then its Poisson cohomology groups $HP^*(S)$ is also a Gerstenhaber algebra via the wedge product and the Schouten-Nijenhuis bracket. Poisson cohomology also acts on Poisson homology via the the cap product and the Lie derivative, which together with the usual de Rham differential over Poisson homology form a differential calculus. For the details, see, for example, [9].

A *Batalin–Vilkovisky algebra* (BV algebra for short) is a Gerstenhaber algebra $(\mathcal{H}^*, \cup, [\ ,\ ])$ together with an operator $\Delta\colon \mathcal{H}^* \to \mathcal{H}^{*-1}$ of degree -1 such that $\Delta \circ \Delta = 0$ and

$$[\alpha, \beta] = (-1)^{|\alpha|}(\Delta(\alpha \cup \beta) - \Delta(\alpha) \cup \beta - (-1)^{|\alpha|} \alpha \cup \Delta(\beta)),$$

for homogeneous elements $\alpha, \beta \in \mathcal{H}^*$.

Batalin-Vilkovisky structure first appeared in mathematical physics. It became interesting for researchers in noncommutative differential geometry (see for example [16]). It seems that interests of algebraists came from a result of T. Tradler ([13]), which in turn was motivated by the string topology invented by M. Chas and D. Sullivan [1]. T. Tradler found that the Hochschild cohomology algebra of a finite dimensional symmetric algebra, such as a group algebra of a finite group, is a BV algebra [13]. L. Menichi [11] gave an independent proof using the language of cyclic operads with multiplications. For precise statements, see the next section.

2. BV structures in Hochschild cohomology

Theorem 1. [13][11] *Let A be a finite-dimensional symmetric algebra or more generally an A_∞ algebra with infinite inner product. Then the Hochschild cohomology of A is a Batalin-Vilkovisky algebra.*

Theorem 2. [6] *Let A be a Calabi-Yau algebra. Then its Hochschild cohomology $HH^*(A)$ is a Batalin-Vilkovisky algebra.*

The proofs of the above results have the same pattern. That is, under the assumption that A is a symmetric algebra or a Calabi-Yau algebra, there exists a paring or a duality between Hochschild cohomology and Hochschild homology, then the Connes' differential on Hochschild homology induces an operator on Hochschild cohomology, which is exactly the BV operator desired.

Inspired by these results, T. Lambre introduced in [8] the notion of a differential calculus with duality. Roughly speaking, this means that there exists a fundamental class in a certain homology group such that the cap product with it establishes an isomorphism between cohomology groups and homology groups. This notion explains when BV structure exists and unifies the two known cases of symmetric algebras and Calabi-Yau algebras.

In [6], the author raised a question, which he attributed to R. Rouquier. We proved this conjecture with X. Chen and S. Yang.

Theorem 3. [2] *Let A be a Koszul Calabi-Yau algebra, and let $A^!$ be its Koszul dual algebra (which is necessarily a graded symmetric algebra). Then there is an isomorphism*

$$\mathrm{HH}^\bullet(A;A) \simeq \mathrm{HH}^\bullet(A^!;A^!)$$

of Batalin-Vilkovisky algebras between the Hochschild cohomology of A and $A^!$.

Recently as an application of this notion, N. Kowalzig and U. Krähmer proved the following result, generalizing the result of Ginzburg.

Theorem 4. [7] *The Hochschild cohomology ring of a twisted Calabi-Yau algebra is also a Batalin-Vilkovisky algebra, provided the Nakayama automorphism (or the modular automorphism) is semisimple.*

Together with T. Lambre and A. Zimmermann and independently by Y. Volkov [14], we showed an analogous result for Frobenius algebras, generalising the result of Tradler-Menichi.

Theorem 5. [10] *Let A be a Frobenius algebra with semisimple Nakayama automorphism. Then the Hochschild cohomology ring $HH^*(A)$ of A is a Batalin-Vilkovisky algebra.*

Note that this result can be deduced from Kowalzig-Krähmer's result via Koszul duality.

3. BV structures in Poisson cohomology

In the nineties, P. Xu already showed that the Poisson cohomology of a unimodular Poisson manifold is a BV algebra [16]. It is folklore that this result holds for smooth affine Poisson algebras.

Theorem 6. *Let S be a smooth affine Poisson algebra which is unimodular. Then the Poisson cohomology $HP^*(S)$ is a BV algebra.*

The proof can be done following the idea of differential calculi with duality. In fact, there is a Poincaré duality between Poisson cohomology and Poisson homology for a unimodular smooth affine Poisson algebra and the de Rham differential on Poisson cohomology induces the BV operator on Poisson cohomology.

For a smooth affine Poisson algebra whose canonical bundle is trivial, there is a well known notion of modular derivation. When this modular derivation vanishes (up to a log-Hamiltonian derivation), this Poisson algebra is unimodular. Even if the Poisson algebra is not unimodular, there exists still a twisted Poincaré duality between Poisson cohomogy and Poisson homology with coefficient in a certain Poisson module. One should be able to show that when the modular derivation is diagonalisable, Poisson cohomology is still a BV algebra.

Now we return to Frobenius Poisson algebras. In a recent paper [15] joint with S.-Q. Wang, Q.-S.Wu and C. Zhu, we were able to define the notion of modular derivation. C. Zhu, F. Van Oystayen and Y.-H. Zhang [17] proved the existence of BV structure on the Poisson cohomology of a unimodular Frobenius Poisson algebra. We generalized their result to the case when the modular derivation is diagonalisable and we also provided examples to show that the condition to be diagonalizable is necessary [15].

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Proper smooth local DG algebras are trivial

Yang Han

Assume that k is a field and all algebras are associative k-algebras with identity. In 1954, Eilenberg proved that if a finite dimensional algebra A is of finite global dimension then its Cartan determinant $\det C(A)$ equals to ± 1 (Ref. [2]). Thereafter, the following conjecture was posed:

Cartan determinant conjecture. Let A be an artin algebra of finite global dimension. Then $\det C(A) = 1$.

The Cartan determinant conjecture remains open except for some special classes of artin algebras. For finite dimensional algebras, the relations between n-recollements of derived categories of algebras and Cartan determinants of algebras were clarified, and the Cartan determinant conjecture was reduced to derived simple algebras [7]. Therefore, the Cartan determinant conjecture would be proved inductively for all finite dimensional algebras if the following two statements hold:

(1) If a finite dimensional algebra A is of finite global dimension then there is an almost complete idempotent e in A such that $\mathrm{gl.dim} eAe < \infty$. Here, an idempotent e in A is said to be *almost complete* if there is a complete set of orthogonal primitive idempotent $\{e_1, \cdots, e_n\}$ in A such that $e = \sum_{i=2}^{n} e_i$.

(2) If A is a finite dimensional algebra of finite global dimension and e is an almost complete idempotent in A such that $\mathrm{gl.dim} eAe < \infty$ then the derived category $D(A)$ of A admits a recollement relative to $D(k)$ and $D(eAe)$.

In general, the statement (1) is not true though it is right for lots of algebras. Indeed, the derived simple two-point algebras of finite global dimension [3, 6] are counterexamples. However, the statement (2) is always true for at least finite dimensional elementary algebras.

Theorem 1. *Let A be a finite dimensional elementary algebra of finite global dimension and e an almost complete idempotent in A such that* $\mathrm{gl.dim} eAe < \infty$. *Then $D(A)$ admits a recollement relative to $D(k)$ and $D(eAe)$. Furthermore,* $\det C(A) = \det C(eAe)$.

For this we need to consider the DG algebras associated with simple modules. Let A be a finite dimensional elementary algebra with Jacobson radical J, and $e_1, \cdots, e_n$ a complete set of orthogonal primitive idempotents in A. It follows from [1, Theorem 4.3] that the projective module eA induces the following recollement of derived categories of DG algebras:

$$D(E) \overset{\overset{-\otimes^L_A E_E}{\longleftarrow}}{\underset{\mathrm{RHom}_A(E,-)}{\overset{-\otimes^L_E E_A}{\longrightarrow}}} D(A) \overset{\overset{-\otimes^L_{eAe} eA}{\longleftarrow}}{\underset{\mathrm{RHom}_{eAe}(Ae,-)}{\overset{-\otimes^L_A Ae}{\longrightarrow}}} D(eAe)$$

for a DG algebra E and a homological epimorphism of DG algebras $A \to E$. The good truncation $B := \tau_{\leq 0}(E)$ of E is a DG subalgebra of E and quasi-isomorphic to E. The DG algebra B is uniquely determined up to derived equivalence by the idempotent e or $1-e$, or the simple module $S := (1-e)A/(1-e)J$, and called the *DG algebra associated with the simple module* S.

Recall that a DG algebra A is said to be *local* if it satisfies: (1) A is *non-positive*, i.e., $A^i = 0$ for all $i > 0$; (2) $H^0(A)$ is a right Noether local k-algebra with the maximal ideal $\mathfrak{n}$ and the residue field k; (3) $H^i(A)$ is a finitely generated right $H^0(A)$-module for all $i \leq 0$.

Proposition 2. *The DG algebra B associated with the simple module S is local.*

Recall that a DG algebra A is said to be *proper* or *compact* if its total cohomology $H^*(A) = \coprod_{i\in\mathbb{Z}} H^i(A)$ is finite dimensional.

Proposition 3. *The DG algebra B associated with the simple module S is proper if and only if $\mathrm{Tor}^{eAe}_*(Ae, eA) := \coprod_{i=0}^{\infty} \mathrm{Tor}^{eAe}_i(Ae, eA)$ is finite dimensional, i.e., $\mathrm{Tor}^{eAe}_i(Ae, eA) = 0$ for $i \gg 0$.*

Recall that a DG algebra A is said to be *smooth* or *homologically smooth* if A is compact in $D(A^e)$, where $A^e := A^{\mathrm{op}} \otimes_k A$ is the enveloping DG algebra of A.

Proposition 4. *The DG algebra B associated with the simple module S is smooth if and only if the Yoneda algebra $\mathrm{Ext}^*_A(S, S)$ of S is finite dimensional.*

Thanks to these propositions, Theorem 1 can be deduced from the following theorem:

Theorem 5. *A proper smooth local DG algebra B is quasi-isomorphic to k.*

Proof. Step 1. By Keller's cyclic functors [5], we can prove that $HH_i(B) \cong HH_i(k)$ for all $i \in \mathbb{Z}$.

Step 2. We can show that $HH_0(H^0(B)) \cong HH_0(B)$. Thus $HH_0(H^0(B)) \cong HH_0(B) \cong HH_0(k) \cong k$ by Step 1.

Step 3. If C is a finite dimensional elementary local algebra and $HH_0(C) \cong k$ then $C \cong k$. Thus $H^0(B) \cong k$ by Step 2.

Step 4. By Jørgensen's amplitude inequality [4], we can show that $\mathrm{amp}B = 0$, i.e., B admits nonzero cohomology only on degree zero. Therefore, B is quasi-isomorphic to the good truncation $\tau_{\geq 0}(B) \cong H^0(B) \cong k$ as DG algebras by Step 3. □

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Hochschild cohomology of ring objects

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(joint work with Magnus Hellstrøm-Finnsen)

This is a report on recent work by my PhD student Magnus Hellstrøm-Finnsen.

Let $(\mathcal{C}, \wedge, I, \alpha, \lambda, \rho)$ be a monoidal category. Thus $\mathcal{C}$ is a category equipped with a bifunctor

$$\begin{array}{rcl} \wedge\colon \mathcal{C} \times \mathcal{C} & \longrightarrow & \mathcal{C} \\ (A, B) & \mapsto & A \wedge B \end{array}$$

called the *smash product*, and a *unit object* I for this product. The *associator* α and left and right *unitors* λ and ρ are all natural isomorphisms with components

$$\begin{array}{rcl} \alpha_{A,B,C}\colon (A \wedge B) \wedge C & \longrightarrow & A \wedge (B \wedge C) \\ \lambda_A\colon I \wedge A & \longrightarrow & A \\ \rho_A\colon A \wedge I & \longrightarrow & A, \end{array}$$

and these are related by two (classes of) commutative diagrams: the *pentagon diagram* and the *triangle diagram*.

A *ring object* (or *monoid*) in $\mathcal{C}$ is a triple (R, μ, e), where R is an object and

$$\begin{array}{rcl} \mu\colon R \wedge R & \longrightarrow & R \\ e\colon I & \longrightarrow & R \end{array}$$

are morphisms in $\mathcal{C}$, called the *multiplication* and the *unit*. These are subject to the following commutative diagrams:

$$\begin{array}{ccccc}
(R\wedge R)\wedge R & & \xrightarrow{\quad\alpha\quad} & & R\wedge(R\wedge R) \\
\downarrow{\scriptstyle \mu\wedge 1} & & & & \downarrow{\scriptstyle 1\wedge\mu} \\
R\wedge R & \xrightarrow{\mu} & R & \xleftarrow{\mu} & R\wedge R
\end{array}$$

$$\begin{array}{ccccccccc}
R & \xrightarrow{\lambda^{-1}} & I\wedge R & \xrightarrow{e\wedge 1} & R\wedge R & \xleftarrow{1\wedge e} & R\wedge I & \xleftarrow{\rho^{-1}} & R \\
& \searrow^{1} & & & \downarrow{\scriptstyle \mu} & & & \swarrow_{1} & \\
& & & & R & & & &
\end{array}$$

We now follow Hochschild's original approach from [4], and define the Hochschild complex and cohomology for a ring object in a monoidal category enriched over abelian groups.

Definition 1. *Let $(\mathcal{C},\wedge,I,\alpha,\lambda,\rho)$ be a monoidal category enriched over abelian groups, and (R,μ,e) a ring object.*

(1) The Hochschild complex *of R is the sequence*

$$\mathbb{H}^R\colon \operatorname{Hom}_{\mathcal{C}}(I,R) \xrightarrow{d^0} \operatorname{Hom}_{\mathcal{C}}(R,R) \xrightarrow{d^1} \operatorname{Hom}_{\mathcal{C}}(R^{\wedge 2},R) \xrightarrow{d^2} \operatorname{Hom}_{\mathcal{C}}(R^{\wedge 3},R) \xrightarrow{d^3} \cdots$$

of abelian groups, with maps given by

$$\begin{aligned}
d^0 f &= \big(\mu\circ(f\wedge 1)\circ\lambda^{-1}\big) - \big(\mu\circ(1\wedge f)\circ\rho^{-1}\big) \\
d^n f &= (\mu\circ(1\wedge f)\circ\alpha) + \sum_{i=1}^{n}(-1)^i\left(f\circ\alpha^{-1}\circ\mu_i^n\circ\alpha\right) + (-1)^{n+1}\left(\mu\circ(f\wedge 1)\right).
\end{aligned}$$

(2) The Hochschild cohomology of R *is defined as the cohomology of the Hochschild complex $\mathbb{H}^R$, i.e.*

$$\mathrm{HH}^n(R) = \mathrm{H}^n\left(\mathbb{H}^R\right) = \operatorname{Ker} d^n / \operatorname{Im} d^{n-1},$$

where d^{-1} is the zero map $0 \longrightarrow \operatorname{Hom}_{\mathcal{C}}(I,R)$.

Remark 2. *(1) The proof showing that the sequence $\mathbb{H}^R$ is a complex involves pairing terms together in such a way that they cancel each other.*

(2) In the definition of the map d^n, the associators α and α^{-1} appearing are in general compositions of the components of the associator in $\mathcal{C}$. Modulo these associators, the map $\mu_i^n\colon R^{\wedge(n+1)} \longrightarrow R^{\wedge n}$ is a smash product of the form $1\wedge\cdots\wedge\mu\wedge\cdots\wedge 1$, involving the multiplication μ once.

(3) There is related work by Ardizzoni, Menini and Ştefan in [1]*, where they define Hochschild cohomology of ring objects in abelian monoidal categories by constructing "bimodule resolutions".*

As in the classical case of Hochschild cohomology of algebras, one can interpret the low dimensional groups $\mathrm{HH}^0(R), \mathrm{HH}^1(R)$ and $\mathrm{HH}^2(R)$ in terms of central elements, derivations and extensions. For instance, the kernel of the map d^1 consists of all maps $f \in \mathrm{Hom}_{\mathcal{C}}(R, R)$ satisfying the equality

$$f \circ \mu = (\mu \circ (1 \wedge f)) + (\mu \circ (f \wedge 1)),$$

and these should be thought of as the derivations of R. The image of d^0 should be thought of as the set of inner derivations of R, so that $\mathrm{HH}^1(R)$ becomes the group of outer derivations on R.

As in the classical case, the Hochschild cohomology of a ring object (R, μ, e) admits a cup product. Take two cocycles

$$f \in \mathrm{Ker}\, d^m \subseteq \mathrm{Hom}_{\mathcal{C}}(R^{\wedge m}, R),$$
$$g \in \mathrm{Ker}\, d^n \subseteq \mathrm{Hom}_{\mathcal{C}}(R^{\wedge n}, R),$$

and define $f \smile g$ as the following compositions:

$$f \smile g\colon \begin{cases} R^{\wedge(m+n)} \xrightarrow{\alpha} R^{\wedge m} \wedge R^{\wedge n} \xrightarrow{f\wedge g} R \wedge R \xrightarrow{\mu} R & \text{if } m, n \geq 1, \\ R^{\wedge m} \xrightarrow{\rho^{-1}} R^{\wedge m} \wedge I \xrightarrow{f\wedge g} R \wedge R \xrightarrow{\mu} R & \text{if } m \geq 1, n = 0, \\ R^{\wedge n} \xrightarrow{\lambda^{-1}} I \wedge R^{\wedge m} \xrightarrow{f\wedge g} R \wedge R \xrightarrow{\mu} R & \text{if } m = 0, n \geq 1, \\ I \xrightarrow{\rho^{-1}=\lambda^{-1}} I \wedge I \xrightarrow{f\wedge g} R \wedge R \xrightarrow{\mu} R & \text{if } m = n = 0. \end{cases}$$

This is again a cocycle, i.e. $f \smile g$ is an element of

$$\mathrm{Ker}\, d^{m+n} \subseteq \mathrm{Hom}_{\mathcal{C}}(R^{\wedge(m+n)}, R).$$

Moreover, we obtain a well defined cup product

$$\smile\colon \mathrm{HH}^m(R) \otimes_{\mathbb{Z}} \mathrm{HH}^n(R) \longrightarrow \mathrm{HH}^{m+n}(R),$$

giving the Hochschild cohomology $\mathrm{HH}^*(R) = \oplus_{n=0}^{\infty} \mathrm{HH}^n(R)$ of R the structure of a graded ring. The identity element of $\mathrm{HH}^*(R)$ is the unit $e\colon I \longrightarrow R$ of R.

Theorem 1 (Hellstrøm-Finnsen, 2016)**.** *If (R, μ, e) is a ring object in a monoidal category $(\mathcal{C}, \wedge, I, \alpha, \lambda, \rho)$ enriched over abelian groups, then its Hochschild cohomology ring $\mathrm{HH}^*(R)$ is graded-commutative with the cup product.*

The proof uses the notions of right pre-Lie systems and pre-Lie rings from [2] and [3]

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The Lie bracket in Hochschild cohomology via the homotopy category of projective bimodules

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(joint work with Johan Steen)

1. The Lie bracket in Hochschild cohomology

Setup. Throughout, we let K be a commutative ring and A a unital and associative K-algebra, which will be assumed to be projective as K-module. We let $A^{\mathrm{ev}} = A \otimes_K A^{\mathrm{op}}$ be the enveloping algebra of A over K, and $\mathbf{K}^-_{A^{\mathrm{ev}}}$ be the homotopy category $\mathbf{K}^-(\mathrm{Proj}(A^{\mathrm{ev}}))$ of bounded below complexes of projective bimodules. Recall that $\mathbf{K}^-_{A^{\mathrm{ev}}} \cong \mathbf{D}^-(A^{\mathrm{ev}})$ by taking "projective resolutions". If C is a category, $\mathsf{C}(X, Y)$ will denote the set of morphisms between $X, Y \in \mathrm{Ob}\,\mathsf{C}$.

1.1. **Background.** In 1963, Murray Gerstenhaber discovered a graded Lie bracket in Hochschild cohomology which governs not only the deformations of the underlying algebra, but also the possible Poisson structures on its center (see [2, 3, 13]). Whereas the multiplicative structure of the Hochschild cohomology algebra can be understood by numerous means, the Lie bracket seems to be a less transparent additional piece of structure. In current work in progress (see [7]) we address the following questions that naturally arise in this context:

(1) Given a projective resolution $P \to A \to 0$ of A over A^{ev}, how can one express the Lie bracket $\{-,-\}_A$ on $\mathrm{HH}^\bullet(A)$ in terms of P? (Classically, the Lie bracket is defined in terms of the bar resolution $B(A)$.)
(2) How can one (intrinsically) define a bimap on $\mathbf{D}(A^{\mathrm{ev}})(A, \Sigma^\bullet A)$ which identifies with $\{-,-\}_A$ along the isomorphism $\mathrm{HH}^\bullet(A) \xrightarrow{\sim} \mathbf{D}(A^{\mathrm{ev}})(A, \Sigma^\bullet A)$?

Answers can, as we will demonstrate, be obtained from what is frequently considered a major weakness of the theory of triangulated categories, namely the non-functoriality of cones.

1.2. **Loop bracket.** An important step towards a solution of the above problems was made by Stefan Schwede in [12], where he described the Lie bracket in Hochschild cohomology in terms of bimodule extensions. In fact, his interpretation made clear that the bracket somewhat reflects the ambiguity of choosing representatives of products of elements in $\mathrm{HH}^n(A) = \mathrm{Ext}^n_{A^{\mathrm{ev}}}(A, A)$. More precisely, Schwede took advantage of the monoidal structure of $(\mathrm{Mod}(A^{\mathrm{ev}}), \otimes_A, A)$ to produce, for given m- and n-self extensions S and T of A with Yoneda composite $S \circ T$, a loop

$$\Omega(S,T) \quad \equiv \quad \begin{array}{ccc} & S \boxtimes_A T & \\ \swarrow & & \searrow \\ S \circ T & & (-1)^{mn} T \circ S \\ \nwarrow & & \nearrow \\ & (-1)^{mn} T \boxtimes_A S & \end{array}$$

in the category $\mathcal{E}xt^{m+n}_{A^{\mathrm{ev}}}(A,A)$ of $(m+n)$-self extensions of A over A^{ev}, that is, an element in the fundamental group, see [10], $\pi_1(\mathcal{E}xt^{m+n}_{A^{\mathrm{ev}}}(A,A), S\circ T)$. This loop identifies with an element in $\mathrm{Ext}^{m+n-1}_{A^{\mathrm{ev}}}(A,A)$ thanks to Vladimir Retakh (see [9] and [11]) who gave an isomorphism

$$\mathrm{Ext}^{n-1}_R(U,V) \xrightarrow{\sim} \pi_1(\mathcal{E}xt^n_R(U,V), S') \quad \text{(for a ring } R \text{ and } U,V \in \mathrm{Mod}(R))$$

which is, in an appropriate sense, independent of the taken base point S'. Schwede's main theorem in this context is now the following.

Theorem 1 (see [12, Thm. 3.1])**.** *Let $m,n \geqslant 1$ be integers. Then for all homogeneous elements $a \in \mathrm{HH}^m(A)$ and $b \in \mathrm{HH}^n(A)$, represented by extensions $S = S(a)$ and $T = T(b)$ respectively, the Gerstenhaber bracket $\{a,b\}_A$ of a and b identifies with the image of the loop $\Omega(S,T)$ in $\mathrm{Ext}^{m+n-1}_{A^{\mathrm{ev}}}(A,A)$.*

Schwede's construction can be extended to the much broader context of exact monoidal categories (see [4, 6]) making it, to some extent, functorial with respect to monoidal functors. This turned out to be a powerful tool enabling the study of the bracket by means of homological algebra (see for instance [5]).

2. The main result

2.1. **Verdier's Lemma.** Recall that Verdier's classical 4×4-Lemma asserts that a given commutative square in a triangulated category $(\mathsf{T},\Sigma_{\mathsf{T}})$ can be completed to a grid of 8 distinguished triangles containing it. Here, we state a slightly modified version of this lemma.

(†)
$$\begin{array}{ccc} X'' & \xrightarrow{f''} & Y'' \\ \downarrow & & \downarrow \\ \Sigma_{\mathsf{T}}X & \xrightarrow{\Sigma_{\mathsf{T}}f} & \Sigma_{\mathsf{T}}Y \end{array} \quad \overset{\text{complete}}{\rightsquigarrow} \quad \begin{array}{ccccccc} X & \longrightarrow & Y & \longrightarrow & Z & \longrightarrow & \Sigma_{\mathsf{T}}X \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ X' & \longrightarrow & Y' & \longrightarrow & Z' & \longrightarrow & \Sigma_{\mathsf{T}}X' \\ \downarrow & & \downarrow & & \downarrow c & & \downarrow \\ X'' & \xrightarrow{f''} & Y'' & \longrightarrow & Z'' & \longrightarrow & \Sigma_{\mathsf{T}}X'' \\ \downarrow & & \downarrow & & \downarrow & -1 & \downarrow \\ \Sigma_{\mathsf{T}}X & \xrightarrow{\Sigma_{\mathsf{T}}f} & \Sigma_{\mathsf{T}}Y & \longrightarrow & \Sigma_{\mathsf{T}}Z & \longrightarrow & \Sigma^2_{\mathsf{T}}X \end{array}$$

All the squares in the completed diagram commute, except the one in the lower right hand corner which anti-commutes. Notice that this completion process is not unique in general, however, all the objects in the 4×4-diagram are, up to (non-canonical) isomorphisms in T, uniquely defined. Special attention will be given to the morphism c later on.

2.2. **The fundamental group of a morphism.** Let $(\mathsf{T},\Sigma_{\mathsf{T}})$ be a triangulated category as before. In [1], Ragnar-Olaf Buchweitz introduced the *fundamental group of a morphism* in T aiming for an adequate triangulated analogue of fundamental groups of extension categories. Formally, for a morphism $\alpha\colon X \to \Sigma_{\mathsf{T}}Y$

in T and $\mathtt{t}$ some distinguished triangle $Y \xrightarrow{i^\alpha} E \xrightarrow{p_\alpha} X \xrightarrow{\alpha} \Sigma_{\mathsf{T}} Y$ containing it, the corresponding fundamental group is defined as the subgroup

$$\pi_1(\alpha, \mathtt{t}) \subseteq \mathrm{Aut}_{\mathsf{T}}(E)$$

of all automorphisms $\chi\colon E \to E$ satisfying $\chi \circ i^\alpha = i^\alpha$ and $p_\alpha \circ \chi = p_\alpha$ (that is, all automorphisms that give rise to an endomorphism of the triangle $\mathtt{t}$). In the same way as the fundamental group of an extension category measures the liberty of choosing representatives of elements in Ext-groups, it resembles the fact that taking cones of morphisms is, in general, a highly non-unique process. Indeed, one of Buchweitz' main observations is the existence of canonical isomorphisms

$$\exp(-i^\alpha_* p^*_\alpha)\colon \mathsf{T}(X,Y) \xrightarrow{\sim} \pi_1(\alpha, \mathtt{t})\,,$$

where X and Y are objects in T satisfying suitable Hom-vanishing conditions (being fulfilled for, e.g., stalk complexes in the derived category), and $i^\alpha_* = \mathsf{T}(E, i^\alpha)$, $p^*_\alpha = \mathsf{T}(p_\alpha, Y)$. The fundamental group of a morphism will play a key role in the considerations below.

2.3. **A sketch of the construction.** In the following, we will give an idea on how to define a bracket operation $[-,-]_A\colon \mathbf{K}^-_{A^{\mathrm{ev}}}(P, \Sigma^m P) \times \mathbf{K}^-_{A^{\mathrm{ev}}}(P, \Sigma^n P) \to \mathbf{K}^-_{A^{\mathrm{ev}}}(P, \Sigma^{m+n-1} P)$. To begin with, we complete four very specific commutative squares to four 4×4-diagrams à la Verdier. Given two morphisms $\alpha\colon P \to \Sigma^m P$ and $\beta\colon P \to \Sigma^n P$, we consider the associated "standard triangles" $\Sigma^{m-1}P \xrightarrow{i^\alpha} E_\alpha \xrightarrow{p_\alpha} P \xrightarrow{\alpha} \Sigma^m P$ and $\Sigma^{n-1}P \xrightarrow{i^\beta} E_\beta \xrightarrow{p_\beta} P \xrightarrow{\beta} \Sigma^n P$. Denoting the tensor product on $\mathbf{K}^-_{A^{\mathrm{ev}}}$ by $\otimes$ (which is extending $\otimes_A$ to complexes), the triangles give rise to the commutative squares in $\mathbf{K}^-_{A^{\mathrm{ev}}}$ below.

$$\begin{array}{ccc} E_\alpha \otimes \Sigma^{n-1}P & \xrightarrow{p_\alpha \otimes i^\beta} & P \otimes E_\beta \\ {\scriptstyle 0}\downarrow & \square_1 & \downarrow{\scriptstyle \alpha \otimes E_\beta} \\ \Sigma^m A \otimes E_\beta & = & \Sigma^m P \otimes E_\beta \end{array} \qquad \begin{array}{ccc} \Sigma^{n-1}P \otimes E_\alpha & \xrightarrow{i^\beta \otimes p_\alpha} & E_\beta \otimes P \\ {\scriptstyle 0}\downarrow & \square_2 & \downarrow{\scriptstyle E_\beta \otimes \alpha} \\ E_\beta \otimes \Sigma^m A & = & E_\beta \otimes \Sigma^m P \end{array}$$

$$\begin{array}{ccc} E_\beta \otimes \Sigma^{m-1}P & \xrightarrow{p_\beta \otimes i^\alpha} & P \otimes E_\alpha \\ {\scriptstyle 0}\downarrow & \square_3 & \downarrow{\scriptstyle \beta \otimes E_\alpha} \\ \Sigma^n P \otimes E_\alpha & = & \Sigma^n P \otimes E_\alpha \end{array} \qquad \begin{array}{ccc} \Sigma^{m-1}P \otimes E_\beta & \xrightarrow{i^\alpha \otimes p_\beta} & E_\alpha \otimes P \\ {\scriptstyle 0}\downarrow & \square_4 & \downarrow{\scriptstyle E_\alpha \otimes \beta} \\ E_\alpha \otimes \Sigma^n P & = & E_\alpha \otimes \Sigma^n P \end{array}$$

For each of these squares $\square_i$, we will obtain a (in general non-unique) isomorphism $c = c(\square_i)$ (as indicated in (†)). The upshot will be, that the four morphisms $c(\square_i)$, for $i = 1, \ldots, 4$, can very naturally be modified to be composable isomorphisms. Their composite will constitute a morphism $\chi(\alpha, \beta)$ in $\pi_1(\gamma, \mathtt{t}(\gamma))$, for $\gamma = (\Sigma^{|\beta|}\alpha) \circ \beta$ and $\mathtt{t}(\gamma)$ the corresponding standard triangle, which will lead us to define

$$[-,-]_A\colon \mathbf{K}^-_{A^{\mathrm{ev}}}(P, \Sigma^m P) \times \mathbf{K}^-_{A^{\mathrm{ev}}}(P, \Sigma^n P) \longrightarrow \mathbf{K}^-_{A^{\mathrm{ev}}}(P, \Sigma^{m+n-1} P)$$

by $[\alpha, \beta]_A = \exp(-u^\gamma_* v^*_\gamma)^{-1}(\chi(\alpha, \beta))$. We now prove the following.

Theorem 2. *Under the canonical isomorphism*

$$\mathrm{HH}^\bullet(A) = \mathrm{Ext}^\bullet_{A^{\mathrm{ev}}}(A, A) \xrightarrow{\sim} \mathbf{K}^-_{A^{\mathrm{ev}}}(P, \Sigma^\bullet P)\,,$$

the Gerstenhaber bracket $\{-,-\}_A$ *on* $\mathrm{HH}^\bullet(A)$ *is taken to the bracket* $[-,-]_A$ *on* $\mathbf{K}^-_{A^{\mathrm{ev}}}(A, \Sigma^\bullet A)$ *described above.*

The proof of Theorem 2 makes use of the canonical functor $\mathcal{E}xt^n_R(U,V) \to \mathbf{D}(R)$ given by double truncation, that is, by mapping an n-extension $0 \to V \to \mathbb{E} \to U \to 0$ to the middle term complex $\mathbb{E}$ concentrated in (homological) degrees 0 up to $n-1$. This functor turns morphisms into isomorphisms and thus gives rise to a group homomorphism $\pi_1(\mathcal{E}xt^n_R(U,V), S) \to \mathrm{Aut}_{\mathbf{D}(R)}(\mathbb{E})$ for each n-extension S with middle term complex $\mathbb{E}$, which factors through the fundamental group of the morphisms $\alpha(S)\colon U \to \Sigma^n V$ corresponding to the equivalence class of S in $\mathrm{Ext}^n_R(U,V)$.

2.4. **Consequences and perspectives.** Intriguing consequences of the theorem are the derived invariance of the bracket (cf. [8]), even better, that any monoidal and triangulated functor will induce a homomorphism of Gerstenhaber algebras. We expect that from here there is a lot more to gain, such as an interplay of recollements of derived categories and the Gerstenhaber bracket in Hochschild cohomology. This direction, however, remains to be investigated.

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Hochschild cohomology of monomial algebras

María Julia Redondo
(joint work with Lucrecia Román)

The aim of this talk is to present some recent results concerning the Gerstenhaber structure of the Hochschild cohomology of a monomial algebra. We use the Bardzell resolution [1] in order to compute it, and we define comparison morphisms between the Bardzell resolution and the standard bar resolution in order to get conditions for the non-vanishing of the cup product and the Lie bracket.

We prove several results in the particular case of string triangular algebras [2] and quadratic string algebras [3]:

(1) we find an explicit description of the generators of the Hochschild cohomology groups;
(2) we find an explicit description of the cup product in even degrees;
(3) if $\operatorname{char} k \neq 2$ we show that $HH^n \cup HH^m = 0$ for any pair of odd natural numbers n, m.

For string triangular algebras we prove that the cup product is always trivial and that the Lie bracket satisfies the equality $[HH^1, HH^n] = HH^n$.

Concerning the non-vanishing of these structures, we consider *gentle cycles*, that is, oriented cycles $\alpha_1 \cdots \alpha_n$ with all possible zero relations of length two and such that for any i, $\alpha_i : x \to y$ is the unique arrow in the quiver ending at y that is involved in a zero relation with α_{i+1} and, it is the unique arrow in the quiver starting at x that is involved in a zero relation with α_{i-1}.

If the quiver associated to the algebra does not contain gentle cycles, we can show that the cup product is trivial for quadratic string algebras and that the Lie bracket is trivial for gentle algebras. On the other hand, we show that these structures are non-trivial in infinitely many degrees if the quiver contains a gentle cycle.

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Quantizations of complete intersection surfaces and D-modules

Travis Schedler

This is a report on a work in progress. The motivation is to understand and ultimately classify quantizations of complete intersections, particularly with isolated singularities.

Jacobian Poisson structure on complete intersection surfaces. Let $f_1, \dots, f_m \in \mathbf{C}[x_1, \dots, x_n]$ be a regular sequence, and suppose $m = n - 2$. Let $X = \{f_1 = \cdots = f_m = 0\}$ be the resulting surface (called a complete intersection surface in $\mathbf{C}^n$). Then, there is a canonical Poisson structure on $\mathbf{C}^n$, given by the bivector $\pi_{\mathrm{Jac}} := (\partial_1 \wedge \cdots \wedge \partial_n)(df_1 \wedge \cdots \wedge df_m)$, where $\partial_i := \frac{d}{dx_i}$. This is parallel to the level surfaces of $f_1, \dots, f_m$, and hence restricts to a Poisson bivector on X itself.

Poisson homology. A Poisson algebra is defined as a commutative algebra equipped with a Poisson bracket. Given such an algebra B, its zeroth Poisson homology, $\mathsf{HP}_0(B)$, is defined by $\mathsf{HP}_0(B) := B/\{B, B\}$.

Theorem 1 ([ES14])**.** *If $B = \mathcal{O}(X)$ for X a complete intersection surface in $\mathbf{C}^n$ with only isolated singularities, then $\mathsf{HP}_0(B) \cong H^2_{\mathrm{top}}(X) \oplus \bigoplus_{x \in X^{\mathrm{sing}}} \mathbf{C}^{\mu_x}$.*

In the theorem above, $H^2_{\mathrm{top}}(X)$ is the topological cohomology of X in the complex topology, X^{sing} is the singular locus, and for each singular point $x \in X^{\mathrm{sing}}$, μ_x denotes the Milnor number of the singularity at x.

General complete intersections. In the case of a general complete intersection X in $\mathbf{C}^n$ (i.e., $X = \{f_1 = \cdots = f_m = 0\}$, with $f_1, \dots, f_m$ a regular sequence for arbitrary m), we do not anymore get a canonical Poisson structure on X, but we do get a canonical polyvector field of degree $n - m$. That is, we get the canonical skew-symmetric multiderivation $M : \mathcal{O}(X)^{\otimes(n-m)} \to \mathcal{O}(X)$, given by $g_1 \otimes \cdots \otimes g_{n-m} \mapsto (\partial_1 \wedge \cdots \wedge \partial_n)(df_1 \wedge \cdots \wedge df_m \wedge dg_1 \wedge \cdots \wedge dg_{n-m})$. Then the theorem above generalizes to:

Theorem 2 ([ES14])**.** *If X is a complete intersection in $\mathbf{C}^n$ with isolated singularities, then $\mathcal{O}(X)/im(M) \cong H^{\dim X}_{\mathrm{top}}(X) \oplus \bigoplus_{x \in X^{\mathrm{sing}}} \mathbf{C}^{\mu_x}$.*

We observe also that, given any additional functions $g_1, \dots, g_{n-m-2}$, we have a Poisson bracket on X given by $\{f, g\}_{g_1,\dots,g_{n-m-2}} := M(g_1, \dots, g_{n-m-2}, f, g)$. This only depends on the exact $n - m - 2$ form $\alpha := dg_1 \wedge \cdots \wedge dg_{n-m-2}$, so let us denote it by $\{-,-\}_\alpha$.

Quantization of complete intersections. By Kontsevich's formality theorem [Kon03] and its sequels, all Poisson structures on a smooth complex affine variety can be quantized. A quantization of a Poisson algebra B (or of a Poisson variety $\mathsf{Spec}\, B$) means an associative product $\star$ on $B[\![\hbar]\!] = \{\sum_{i \geq 0} b_i \hbar^i \mid b_i \in B\}$ such that, for all $a, b \in B$, we have $a \star b - ab \in \hbar B_\hbar$ and $(a \star b - b \star a - \hbar\{a, b\}) \in \hbar^2 B_\hbar$.

Very little is known about the case of singular varieties. However, we can prove the following.

Proposition 3. *Every Poisson structure on a complete intersection of the form $\{-,-\}_\alpha$, for α a closed $n - m - 2$-form, can be quantized.*

Proof. Consider the dg algebra resolution $\mathcal{O}(\mathbf{C}^{n|m}) := \mathbf{C}[x_1, \dots, x_n, r_1, \dots, r_m]$ with $|r_i| = -1$ and $|x_i| = 0$ for all i, equipped with the differential d given by $dx_i = 0$ and $dr_i = f_i$ for all i. The Poisson bracket $\{-,-\}_\alpha$ extends to this

resolution by setting brackets with r_i to be zero. Then, a quantization of this Poisson bracket can be identified with a Maurer-Cartan element in the dg Lie algebra $(D_{\mathrm{poly}}(\mathbf{C}^{n|m}), d + d_H)$. Here d_H is the Hochschild differential, and the differential d is induced from $d(r_i) = f_i$. Observe that d can also be written as the Gerstenhaber bracket with the derivation $D(x_i) = 0, D(r_i) = f_i$ for all i (which is a Maurer-Cartan element).

By Kontsevich's formality theorem [Kon03], there is an L_∞ quasi-isomorphism $T_{\mathrm{poly}}(\mathbf{C}^{n|m}) \to D_{\mathrm{poly}}(\mathbf{C}^{n|m})$. Here, $T_{\mathrm{poly}}[-1] \cong \mathcal{O}(\mathbf{C}^{n|m}) \otimes \mathbf{C}[\partial_{x_1}, \ldots, \partial_{x_n}, \partial_{r_1}, \ldots, \partial_{r_m}]$, for $|\partial_{x_i}| = 1$ and $|\partial_{r_i}| = 2$. The quasi-isomorphism sends $\sum_i f_i \partial_{r_i}$ to D. Therefore by Maurer-Cartan twisting, we obtain an L_∞ quasi-isomorphism $(T_{\mathrm{poly}}(\mathbf{C}^{n|m}), d) \to (D_{\mathrm{poly}}(\mathbf{C}^{n|m}), d + d_H)$, with d_H the Hochschild differential, and in both cases d is the differential obtained from $d(r_i) = f_i$ (which identifies with bracketing with $\sum_i f_i \partial_{r_i}$ and with D, respectively).

To conclude, the Poisson bracket $\{-,-\}_\alpha$ defines a Maurer-Cartan element of $(T_{\mathrm{poly}}(\mathbf{C}^{n|m}), d)$, hence by Kontsevich's theorem, also such an element of $(D_{\mathrm{poly}}(\mathbf{C}^{n|m}), d + d_H)$, which quantizes the resolution. Taking cohomology, we obtain a quantization of X. □

Classification of quantizations and noncommutative Poisson cohomology.

Definition 4. [BG92, Xu94] *A noncommutative Poisson structure on an associative algebra A is an element $\mu \in \mathsf{HH}^2(A, A)$ such that $[\mu, \mu] = 0 \in \mathsf{HH}^3(A, A)$. The noncommutative Poisson cohomology, denoted $\mathsf{HP}^*_{\mathrm{nc}}(A, \mu)$, is the cohomology of $(\mathsf{HH}^*(A, A), [\mu, -])$.*

When $A = \mathcal{O}(X)$ for X a Poisson variety, the above need **not** coincide with usual Poisson cohomology, although this is true when X is smooth affine (where one recovers Lichnerowicz's definition of Poisson cohomology [Lic78]).

Given a noncommutative Poisson structure, we have a corresponding infinitesimal deformation of A which lifts to a second-order deformation. As we explain, the noncommutative Poisson cohomology controls deformations of A whose infinitesimal class is μ. Recall that an n-th order deformation is an element $\star = \mu_A + \hbar\mu_1 + \hbar^2\mu_2 + \cdots + \hbar^n\mu_n \in \hbar C^2(A, A)[\![\hbar]\!]$ whose Gerstenhaber bracket with itself is zero modulo $\hbar^{n+1}$ (here μ_A is the multiplication on A, so $[\mu_A, x] = d_H x$ for all x, which d_H the Hochschild differential). We say this lifts μ if the class of μ_1 is μ. An k-th order gauge equivalence between two n-th order deformations $\star$ and $\star'$ is an element $\gamma \in \hbar^k C^1(A, A)[\![\hbar]\!]$, such that $(1+\gamma)(a \star b) \equiv (1+\gamma)(a) \star' (1+\gamma)(b) \pmod{\hbar^{n+1}}$.

Proposition 5. *Given an n-th order deformation lifting μ, if a further lift to $(n+2)$-nd order exists, the space of lifts to $(n+1)$-st order modulo n-th order gauge equivalences is an affine space on $\mathsf{HP}^2_{\mathrm{nc}}(A, \mu)$.*

This is a variant on the standard result which says that, when a lift of an arbitrary n-th order deformation to $(n+1)$-st order exists, the space of such lifts modulo $(n+1)$-st order gauge equivalences is an affine space over $\mathsf{HH}^2(A, A)$.

Proof. Let $\star = \mu_A + \sum_{i=1}^{n} \hbar^i \mu_i$ be an n-th order deformation lifting μ. Suppose $\star' = \mu_A + \sum_{i=1}^{n+2} \hbar^i \mu_i'$ and $\star'' = \mu_A + \sum_{i=1}^{n+2} \hbar^i \mu_i''$ are further lifts of $\star$ to $(n+2)$-nd order deformations, with $\mu_i = \mu_i' = \mu_i''$ for $i \leq n$. Then the coefficient of $\hbar^{n+2}$ in $[\star', \star'] - [\star'', \star'']$ is $d_H(\mu_{n+2}' - \mu_{n+2}'') + [\mu_1, \mu_{n+1}' - \mu_{n+1}'']$. Also, as is well-known, $\mu_{n+1}' - \mu_{n+1}''$ must be a Hochschild two-cocycle (which follows also from the $\hbar^{n+1}$ coefficient). Therefore, $\mu_{n+1}' - \mu_{n+1}''$ is a Hochschild two-cocycle whose class in $\mathsf{HH}^2(A, A)$ has zero bracket with μ, i.e., defines a noncommutative Poisson two-cocycle.

Now suppose only that $\star'$ is an $n+1$-st order deformation lifting $\star$. Given $\gamma = \sum_{i \geq n} \hbar^i \gamma_i \in \hbar^n C^1(A, A)[\![\hbar]\!]$, let $\star_\gamma'$ be the resulting gauge equivalent $(n+1)$-nd order deformation. Then $\star_\gamma' - \star' \equiv d_H \gamma + \hbar[\gamma, \mu_1] \pmod{\hbar}^{n+2}$. For the result to lift $\star$ we require γ_n to be a two-cocycle, and in this case the coefficient of $\hbar^{n+1}$ in $\star_\gamma' - \star'$ consists of all Hochschild two-cocycles whose cohomology class is a noncommmutative Poisson two-coboundary. □

Poisson cohomology for complete intersections. To compute the Poisson cohomology for complete intersections, we need to compute the Gerstenhaber bracket on Hochschild cohomology. By the proof of Proposition 3, the latter can be computed by $H^*(T_{\text{poly}}(\mathbf{C}^{n|m}), d)$ together with the Schouten bracket on $\mathbf{C}^{n|m}$. In the case of $m = 1, n = 3$, setting $f := f_1$, a direct computation yields the well-known result:

$$\mathsf{HH}^2(\mathcal{O}(X)) \cong T^2_{\text{poly}}(X) \oplus C_X,$$

where we define $T^2_{\text{poly}}(X)$ as the space of skew-symmetric biderivations $\mathcal{O}(X)^{\otimes 2} \to \mathcal{O}(X)$, and $C_X := \mathcal{O}(X)/(\partial_1 f, \partial_2 f, \partial_3 f)$, called the singularity ring. Also, $\mathsf{HH}^1(\mathcal{O}(X) = \text{Der}(\mathcal{O}(X)) = T^1_{\text{poly}}(X)$. Setting $\xi = \xi_g := \{g, -\}$ for some $g \in \mathcal{O}(X)$, we get $[\xi_g, h\pi_{\text{Jac}}] = \{g, h\}\pi_{\text{Jac}}$. This implies:

Proposition 6. *If X is a surface in $\mathbf{C}^3$, then the space $\mathsf{HP}^2_{nc}(\mathcal{O}(X), [\pi_{Jac}])$ is a quotient of $\mathsf{HP}_0(\mathcal{O}(X)) \oplus C_X$.*

Applying Theorem 1, we obtain:

Corollary 7. *If X is a hypersurface in $\mathbf{C}^3$ with isolated singularities, then $\mathsf{HP}^2_{nc}(\mathcal{O}(X), [\pi_{Jac}])$ is a quotient of $C_X \oplus H^2_{\text{top}}(X) \oplus \bigoplus_{x \in X^{\text{sing}}} \mathbf{C}^{\mu_x}$.*

In this case, we expect that a universal family of quantizations exists. Assuming this, by Proposition 5, we obtain:

Corollary 8. *The universal family of quantizations of a surface in $\mathbf{C}^3$ with isolated singularities, having infinitesimal class $[\pi_{Jac}]$, is parameterized by a quotient of $\bigoplus_{m \geq 2} \hbar^m (C_X \oplus H^2_{\text{top}}(X) \oplus \bigoplus_{x \in X^{\text{sing}}} \mathbf{C}^{\mu_x})$.*

Note that the space $\mathsf{HP}^2_{\text{nc}}(\mathcal{O}(X), [\pi_{\text{Jac}}])$ parameterizing lifts from deformations of order n to order $n+1$ is **finite-dimensional**. In the case that π_{Jac} is weighted homogeneous of degree zero (i.e., X is conical with an elliptic singularity), we get that $\mathsf{HP}_{\text{nc}}(X) \cong C_X \oplus \mathbf{C}$, of dimension equal to one more than the Milnor number. In this case, the universal quantization was described in [EG10] in terms

of a quotient of a Calabi-Yau deformation of $\mathcal{O}(\mathbf{C}^3)$ by a central element. In the case π_{Jac} is weighted homogeneous of negative degree, X is a du Val singularity, and we get $\mathsf{HP}_{\mathrm{nc}}(X) \cong C_X$, of dimension equal to the Milnor number; in this case we expect to recover that the universal quantization coincides with the global sections of the universal quantization of the minimal resolution of singularities $\tilde{X}$ (whose base of deformation is the functions on a formal neighborhood of the origin in $H^2_{\mathrm{top}}(\tilde{X})$, also of the same dimension as $\mathsf{HP}^2_{\mathrm{nc}}(\mathcal{O}(X), [\pi_{\mathrm{Jac}}])$).

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